CYCLIC HOMOLOGY OF ALGEBRAS

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To Sebastian, Julie

Anna and Willy

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Introduction.

These lectures are an extended version of my contribution to a seminar on cyclic cohomology, held at the University of Marseille Luminy, in 1985. They are essentially based on a paper of J.L. Loday and D. Quillen: Cyclic homology and the Lie algebra homology of matrices (Comment: Math. Helvetici, 59 (1984), 565-591), and contain also ideas and results of M. Karoubi and C. Kassel.

The exposition is purely algebraic, according to my own background, and thus concentrates rather on cyclic homology (than on cohomology), the former being a more natural starting point for an algebraist. But many of the leading ideas of the theory, more apparent in cohomology, come from topology and differential geometry, in the language of operator algebras. Thus it should be clear that this is not an introduction to cyclic (co)homology, but only the attempt to single out the basic algebraic facts and techniques of the theory. The reader who wants more motivations should imperatively consult the fundamental article of Alain Connes: Noncommutative differential geometry, I.H.E.S. Publ. Math. vol. 62 (1985), 41-144.

The lectures are organized in two chapters.

The first chapter deals with the intimate relation of cyclic theory to ordinary Hochschild theory, which is at least not surprising by the parallel definition of both theories. There are some important quasi-isomorphisms, proving the equivalence of different approaches to cyclic homology, and spectral sequence techniques are convenient to establish these facts. Thus a comforting paragraph on spectral sequences opens the exposition. Fortunately, I could already take in account the extremely elegant mixed complex approach to cyclic homology of D. Burghlelea which streamlines a lot of arguments. The first climax is the fundamental long exact sequence

$$\cdots \rightarrow H_n(A) \rightarrow HC_n(A) \rightarrow HC_{n-2}(A) \rightarrow H_{n-1}(A) \rightarrow \cdots$$

relating the Hochschild homology groups and the cyclic homology groups (analogously in cohomology), a cornerstone for all structural transmission between both theories. Normalized mixed Hochschild complexes and reduced theory are treated in order to invest conveniently differential ideas: Our operator B becomes a good candidate for a non-

commutative outer derivative. Finally, Morita-invariance of Hochschild homology and of cyclic homology are treated, following closely an exposition of K. Igusa.

The second chapter deals with cyclic homology as a typical characteristic zero theory. First, its relation to de Rham cohomology is considered. It comes out that (noncommutative) de Rham cohomology in the sense of M. Karoubi can be embedded in (reduced) cyclic homology. For smooth commutative algebras this can be made more precise by a sort of inverse limit constellation, which is formulated via a decomposition of cyclic homology into ordinary de Rham cohomology:

$$HC_n(A) = \Omega^n/d\Omega^{n-1} \oplus H_{DR}^{n-2}(A) \oplus H_{DR}^{n-4}(A) \oplus \dots$$

This result of J.L. Loday and D. Quillen has a dual version in continuous cyclic cohomology, due to A. Connes (with $A = C^\infty(X, \mathbb{C})$, the $\mathbb{C}$ -algebra of smooth complex-valued functions on a compact manifold X). The final sections of the second chapter deal with cyclic homology as "additive K-theory", in the following sense: For an associative algebra A over a field k of characteristic zero, cyclic homology $HC_*(A)$ is, up to a dimension shift, isomorphic to the space of primitive elements $\text{Prim } H_*(\mathfrak{gl}(A))$ of the Lie algebra homology of $\mathfrak{gl}(A) = \varinjlim \mathfrak{gl}_r(A)$. This result should be appreciated in the light of D. Quillen's "multiplicative" version: Rational algebraic K-theory $K_*(A) \otimes \mathbb{Q}$ identifies with $\text{Prim } H_*(GL(A), \mathbb{Q})$, the primitive part of (discrete) group homology of $GL(A) = \varinjlim GL_r(A)$. I have to admit two important algebraic omissions: First, I did not treat products (essential in cohomology, which I neglected a bit) and Künneth-formulas (since I got afraid of the co-structure invasion). Then, which is perhaps more serious, I did not treat the relations to algebraic K-theory via Chern-characters. On a certain level of arguments this motivates half of the existence of the theory ("create a range for invariants"), but whenever you make the first step towards topology and geometry ...

A few words about the use of spectral sequences in these lectures. We only need them in order to establish some fundamental quasi-isomorphisms, by an approximation argument. This could also be done via explicit and involved calculations. But, since we aim at furnishing the necessary material for further reading, there is no reason to avoid spectral sequence techniques (look at the literature!). At any rate, there is a coherent approach to the basic skeleton of the theory, avoiding all spectral sequence arguments. You begin with I.2.3; then you define cyclic homology of a unital associative k -algebra A as the cyclic homology of the mixed complex $C(A)$ (cf. I.2.4.2). You get all of the

material from I.2.4.6 to I.2.4.12. Reduced theory remains unchanged (I.2.5.1 to I.2.5.13). Cyclic cohomology is treated analogously. As to Morita-invariance, the spectral sequence argument in I.2.7.9 is easily replaced by a direct reasoning. de Rham theory (II.1) is already clean. Thus you cover rapidly all of the basic material. It should be pointed out, however, that the equivalence of the different approaches to cyclic (co)homology is an essential part of its handiness. Finally, I would like to express my thanks to all those who helped me to finish these lectures: first of all, to Daniel Kastler, whose stimulating enthusiasm for the subject and clever support (on many levels) pushed me across this experience. Then, to Joachim Cuntz and Georges Zeller-Meier, who taught me the essentials of the subject, and finally to Philippe Blanchard, Sergio Doplicher, Rudolph Haag and Daniel Testard, whose hospitality and interest at different stages of the work I shall never forget.

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Peter Seibt