UNDERGRADUATE CONVEXITY

From Fourier and Motzkin to Kuhn and Tucker

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From Fourier and Motzkin to Kuhn and Tucker

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Preface

Convexity is a key concept in modern mathematics with rich applications in economics and optimization.

This book is a basic introduction to convexity based on several years of teaching the one-quarter courses *Konvekse Mængder* (convex sets) and *Konvekse Funktioner* (convex functions) to undergraduate students in mathematics, economics and computer science at Aarhus University. The prerequisites are minimal consisting only of first year courses in calculus and linear algebra.

I have attempted to strike a balance between different approaches to convexity in applied and pure mathematics. Compared to the former the mathematics takes a front seat. Compared to some of the latter, a key point is that the ability to carry out computations is considered paramount and a crucial stepping stone to the understanding of abstract concepts e.g., the definition of a face of a convex set does not make much sense before it is viewed in the context of several simple examples and computations.

Chapters 1–6 treat convex subsets from the basics of linear inequalities to Minkowski’s theorem on separation of disjoint convex subsets by hyperplanes. The basic idea has been to emphasize part of the rich “finite” theory of polyhedra before entering into the “infinite” theory of closed convex subsets.

Fourier-Motzkin elimination is to linear inequalities what Gaussian elimination is to linear equations. It seems appropriate to begin a course on convexity by introducing this simple, yet powerful method. The prerequisites are barely present. Still the first chapter contains substantial results such as a simple algorithm for linear optimization and the fundamental theorem that projections of polyhedra are themselves polyhedra.

Before introducing closed convex subsets, several basic definitions and highlights from the polyhedral world are given: a concise treatment of affine subspaces, faces of convex subsets, Bland's rule from the simplex algorithm as a tool for computing with the convex hull, faces of polyhedra, Farkas's lemma, steady states for Markov chains, duality in linear programming, doubly stochastic matrices and the Birkhoff polytope.

The chapter *Computations with polyhedra* contains a treatment of two important polyhedral algorithms: the double description method and the simplex algorithm. The double description method is related to Fourier-Motzkin elimination. It is very easily explained in an undergraduate context especially as a vehicle for computing the bounding half spaces of a convex hull.

The simplex algorithm solves linear optimization problems and is somewhat mysterious from a mathematical perspective. There is no obvious reason it should work well. In fact, the famous mathematician John von Neumann never really believed it would perform in practice. The inventor George B. Dantzig also searched for alternate methods for years before confronting experimental data from some of the world's first computers: the simplex algorithm performed amazingly well in practice. Only recently has a mathematical explanation for this phenomenon been given by Spielman and Teng. Our treatment of the simplex algorithm and the simplex tableau deviates from the standard form and works with the polyhedron in its defining space.

The transition to the "continuous" theory of non-polyhedral convex subsets comes after the first five chapters. Here it is proved that closed convex subsets serve as generalizations of polyhedra, since they coincide with arbitrary intersections of affine half spaces. The existence of a supporting hyperplane at a boundary point of a convex subset is proved and Minkowski's theorems on compact convex subsets and separation of disjoint convex subsets are given.

Chapters 7–10 treat convex functions from the basic theory of convex functions of one variable with Jensen's inequality to the Karush-Kuhn-Tucker conditions, dual optimization problems and an outline of an interior point algorithm for solving convex optimization problems in several variables. The setting is almost always the simplest. Great generality is fine when you have lived with a subject for years, but in an introductory course it tends to become a burden. You accomplish less by including more.

The main emphasis is on differentiable convex functions. Since undergraduate knowledge of differentiability may vary, we give an almost com-

plete review of the theory of differentiability in one and several variables. The only “general” result on convex functions not assuming differentiability is the existence of the subgradient at a point.

An understanding of convex functions of several variables is impossible without knowledge of the finer points of linear algebra over the real numbers. Introducing convex functions of several variables, we also give a thorough review of positive semidefinite matrices and reduction of symmetric matrices. This important part of linear algebra is rarely fully understood at an undergraduate level.

The final chapter treats *Convex optimization*. The key elements are the Karush-Kuhn-Tucker conditions, how saddle points of the Lagrangian lead to a dual optimization problem and finally an outline of an interior point algorithm using bisection and the modified Newton method. Monographs have been written on these three topics. We only give a brief but self-contained introduction with simple examples.

Suggestions for teaching a one-semester course

The amount of material included in this book exceeds a realistic plan for a one-semester undergraduate course on convexity. I consider Fourier-Motzkin elimination (Chapter 1), affine subspaces (Chapter 2), basics of convex subsets (Chapter 3), the foundational material on polyhedra in Chapter 4, a taste of one of the two algorithms in Chapter 5 and closed convex subsets (Chapter 6) as minimum along with almost all of the material in Chapters 7–10.

The progression of learning depends on the proficiency in linear algebra and calculus. The necessary basic concepts from analysis are introduced in Appendix A. In Appendix B there is a review of linear algebra from the point of view of linear equations leading to the rank of a matrix.

In my view, a too rigid focus on the abstract mathematical details before telling about examples and computations is a major setback in the teaching of mathematics at all levels. Certainly the material in this book benefits from being presented in a computational context with lots of examples.

Aarhus, December 2012

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I am an algebraist by training and encountered convexity because of an interest in computational algebra (and computers!). As such, I have benefited immensely over the years from insightful explanations from the following more knowledgeable people: Tage Bai Andersen, Kent Andersen, Kristoffer Arnsfelt Hansen, Peter Bro Miltersen, Marcel Bökstedt, Komei Fukuda, Anders Nedergaard Jensen, Herbert Scarf, Jacob Schach Møller, Andrew du Plessis, Henrik Stetkær, Bernd Sturmfels, Rekha Thomas, Jørgen Tornehave, Jørgen Vesterstrøm and Bent Ørsted.

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