Lecture Notes on Mathematical Olympiad Courses

For Senior Section Vol. 2

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For Senior Section

(In 2 Volumes)

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Preface

Although Mathematical Olympiad competitions are carried out by solving problems, the system of Mathematical Olympiads and the related training courses cannot consist only of problem solving techniques. Strictly speaking, it is a system of mathematical advancing education. To guide students, who are interested in and have the potential to enter the world of Olympiad mathematics, so that their mathematical ability can be promoted efficiently and comprehensively, it is important to improve their mathematical thinking and technical ability in solving mathematical problems.

An excellent student should be able to think flexibly and rigorously. Here, the ability to perform formal logic reasoning is an important basic component. However, it is not the main one. Mathematical thinking also includes other key aspects, such as starting from intuition and entering the essence of the subject, through the processes of prediction, induction, imagination, construction and design to conduct their creative activities. In addition, the ability to convert the concrete to the abstract and vice versa is essential.

Technical ability in solving mathematical problems does not only involve producing accurate and skilled-computations and proofs using the standard methods available, but also the more unconventional, creative techniques.

It is clear that the standard syllabus in mathematical education cannot satisfy the above requirements. Hence the Mathematical Olympiad training books must be self-contained basically.

This book is based on the lecture notes used by the editor in the last 15 years for Olympiad training courses in several schools in Singapore, such as Victoria Junior College, Hwa Chong Institution, Nanyang Girls High School and Dunman High School. Its scope and depth significantly exceeds that of the standard syllabus provided in schools, and introduces many concepts and methods from modern mathematics.

The core of each lecture are the concepts, theories and methods of solving mathematical problems. Examples are then used to explain and enrich the lectures, as well as to indicate the applications of these concepts and methods. A number of questions are included at the end of each lecture for the reader to try. Detailed solutions are provided at the end of book.

The examples given are not very complicated so that the readers can understand them easily. However, many of the practice questions at the end of lectures are taken from actual competitions, which students can use to test themselves. These questions are taken from a range of countries, such as China, Russia, the United States of America and Singapore. In particular, there are many questions from China for those who wish to better understand Mathematical Olympiads there. The questions at the end of each lecture are divided into two parts. Those in Part A are for students to practise, while those in Part B test students' ability to apply their knowledge in solving real competition questions.

Each volume can be used for training courses of several weeks with a few hours per week. The test questions are not considered part of the lectures as students can complete them on their own.

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Abbreviations and Notations

Abbreviations

AHSME	American High School Mathematics Examination
AIME	American Invitational Mathematics Examination
AMC	American Mathematics Examination
APMO	Asia Pacific Mathematics Olympiad
ASUMO	Olympics Mathematical Competitions of All the Soviet Union
AUSTRIA	Austria Mathematical Olympiad
AUSTRALIA	Australia Mathematical Olympiad
BALKAN	Balkan Mathematical Olympiad
BELARUS	Belarus Mathematical Olympiad
BMO	British Mathematical Olympiad
BOSNIA	Bosnia Mathematical Olympiad
BULGARIA	Bulgaria Mathematical Olympiad
CGMO	China Girl's Mathematical Olympiad
CHINA	China Mathematical Competition for Secondary Schools except for CHNMOL
CHNMOL	China Mathematical Competition for Secondary Schools
CHNMO	China Mathematical Olympiad
CMC	China Mathematical Competition (for High Schools)
CMO	Canada Mathematical Olympiad
CNMO	China Northern Mathematical Olympiad
CSMO	China Southeastern Mathematical Olympiad
CROATIA	Croatia Mathematical Olympiad
CWMO	China Western Mathematical Olympiad

CZECH-POLISH-SLOVAK	International Competitions Czech-Polish-Slovak Match
ESTONIA	Estonia Mathematical Olympiad
GERMANY	Germany Mathematical Olympiad
GREECE	Greece Mathematical Olympiad
High-School Mathematics	Journal of “High-School Mathematics”, China.
HONG KONG	Hong Kong Mathematical Olympiad
HUNGARY	Hungary Mathematical Competition
HUNGARY-ISRAEL	Hungary-Israel Binational Mathematical Competition
IMO	International Mathematical Olympiad
INDIA	India Mathematical Olympiad
IRAN	Iran Mathematical Olympiad
IRE	Ireland Mathematical Olympiad
JAPAN	Japan Mathematical Olympiad
KOREA	Korea Mathematical Olympiad
MACEDONIA	Macedonia Mathematical Olympiad
MEDITERRANEAN MO	Mediterranean Mathematics Olympiad
POLAND	Poland Mathematical Olympiad
PUTNAM	Putnam Mathematical Competition
ROMANIA	Romania Mathematical Olympiad
RUSMO	All-Russia Olympics Mathematical Competitions
SAUDI ARABIA	Saudi Arabia Mathematical Olympiad
SERBIA	Serbia Mathematical Olympiad
SLOVENIA	Slovenia Mathematical Olympiad
SMO	Singapore Mathematical Olympiads
SSSMO	Singapore Secondary Schools Mathematical Olympiads
SWE	Sweden Mathematical Olympiads
TAIWAN	Taiwan Mathematical Olympiads
THAILAND	Thailand Mathematical Olympiads
TST	Team Selection Test (including related training tests)
TURKEY	Turkey Mathematical Olympiad
UKRAINE	Ukraine Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
USSR	Union of Soviet Socialist Republics
VIETNAM	Vietnam Mathematical Olympiad

Notations for Numbers, Sets and Logic Relations

$\mathbb{N}$	the set of positive integers (natural numbers)
$\mathbb{N}_0$	the set of non-negative integers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers
$\mathbb{Q}_0^+$	the set of non-negative rational numbers
$\mathbb{R}$	the set of real numbers
$[m, n]$	the lowest common multiple of the integers m and n
(m, n)	the greatest common divisor of the integers m and n
$a \mid b$	a divides b
$ x $	the absolute value of x
$\lfloor x \rfloor$	the greatest integer not greater than x
$\lceil x \rceil$	the smallest integer not less than x
$\{x\}$	the decimal part of x , i.e. $\{x\} = x - \lfloor x \rfloor$
$a \equiv b \pmod{c}$	a is congruent to b modulo c
$\binom{n}{k}$	the binomial coefficient n choose k
$n!$	n factorial, equal to the product $1 \cdot 2 \cdot 3 \cdots n$
$[a, b]$	the closed interval, i.e. all x such that $a \leq x \leq b$
(a, b)	the open interval, i.e. all x such that $a < x < b$
$\Leftrightarrow$	iff, i.e. if and only if
$\Rightarrow$	implies
$A \subset B$	A is a subset of B
$A - B$	the set formed by all the elements in A but not in B
$A \cup B$	the union of the sets A and B
$A \cap B$	the intersection of the sets A and B
$a \in A$	the element a belongs to the set A

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