MATHEMATICAL TOPICS IN NONLINEAR KINETIC THEORY

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To Marvin Shinbrot

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PREFACE

Boltzmann's celebrated equation is certainly the most famous and successful mathematical model to describe the time and space evolution, at a statistical level, of the position and velocity distribution function of particles in a dilute gas. This equation, or class of equations, beautiful and elegant, is a powerful instrument both in the analysis of thermodynamical properties of fluids and in the solution of fluid-dynamical problems in rarefied gas dynamics, which are often relevant in modern technology.

The study of the Boltzmann equation is therefore, for this and several other reasons, a relevant field of mathematical physics. On the other hand, one cannot hide several still unsolved problems:

- The Boltzmann equation is a mathematical model easy to derive intuitively, but rather difficult to justify rigorously.
- The Boltzmann equation is a mathematical model based on some simplifications of physical reality. These simplifications are consistent for a rarefied gas, but need to be carefully re-considered for gases which are even moderately dense.
- Despite Boltzmann's simple derivation of the equation and some relevant qualitative and quantitative fluid-dynamical descriptions, i.e. the H-theorem or the computing of transport coefficients, still no satisfactory proof is available for the solution of the initial value problem without restrictions on initial data.

This book is an attempt to provide some answers to the problems which have been described above. These problems have been and still are today a challenging objective and a hard obstacle for applied mathematicians with interest in this field of mathematical physics. Their solution can also shed some light on the Boltzmann equation as a mathematical model.

It is beyond doubt that mathematical results are of enormous importance in testing theory. Quoting Truesdell and Muncaster from the preface to their celebrated book *Fundamentals of Maxwell's Kinetic Theory of a Simple Monoatomic Gas* (Academic Press, 1980):

to compare conclusions, we must first have them. The only way to get a conclusion from a mathematical theory is by logic, by mathematical steps. Any conclusion gotten otherwise, as for example by "physical intuition", blind teamwork on huge machines, or other gilded guessing is really not a conclusion from the theory. At best, it is itself some other theory, not a consequence of one we study. At worst, it is wrong. Once we recognize that a theory is a mathematical model, we recognize also that only rigorous mathematical conclusions from a theory can be accepted in tests of the justice of that theory.

and further

a proved theorem in a physical theory shows how some part of nature might behave. Mathematical botchery proves no more than the failure of the theorist. Therefore, in physical theory, mathematical rigor is of essence!

Considering that we share this opinion, we came to the decision of developing a mathematical analysis, organized into a unified self-consistent form, referred to several topics in nonlinear kinetic theory which, at present, are available only in different articles. The main attention in the book is paid to the existence and uniqueness of solutions of the initial value problem for the full, nonlinear Boltzmann equation in the space inhomogeneous case.

We should keep in mind that no textbook is available with a complete analysis of the space inhomogeneous problems and the large part of existing books deals only with linear aspects of this theory. The reason why we do not include the analysis of the spatially homogeneous problems is that these results are extensively and carefully dealt with in the already cited book by Truesdell and Muncaster.

Since the book is written in the framework of classical analysis, we also came to the decision not to include the results obtained, essentially by L. Arkeryd, in the framework of non-standard analysis. These results can be found in the book by Alberverio, S. et al., *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press (1986).

Considering that the book is referred to mathematical and physical aspects of the kinetic theory of gases, the reader should already possess some knowledge of the kinetic theory of gases in order to understand the structure of equations. Elementary knowledge of functional analysis will also be necessary to understand the various mathematical proofs provided in the book.

However, whenever the proof will require mathematical methods which are not available in standard textbooks of functional analysis, the method itself will be described in details. This is the case of Chapter II, which contains a detailed description of the so-called Kaniel and Shinbrot iterative scheme, and of Chapter III, which provides a description of linear semigroups theory.

We can now present the content of the book. The first chapter deals with the description of the nonlinear Boltzmann equation as a mathematical model in the kinetic theory of gases, the discussion on its validity, and the mathematical formulation of initial value problems. The first chapter also provides a detailed list of the pertinent bibliography in the field.

The second chapter deals with the analysis of the Cauchy problem for initial conditions tending to zero at infinity in the phase space. Existence, uniqueness and asymptotic behaviour of solutions is proved referring to different decay conditions. These theorems are proved under suitable smallness conditions on initial data. Physical consistency of the smallness conditions is discussed in details and is related to the decay conditions. This chapter also provides the analysis of the H-theorem in the case when solutions to the initial value problem exist.

The third chapter deals with the analysis of the Cauchy problem for initial conditions close to equilibrium. This problem has certainly a wider thermodynamical interest than the one dealt with in Chapter II, even if, unfortunately as we shall see, the proof can be obtained only for very small perturbations. After a discussion of existence, uniqueness and trend to equilibrium of the solutions to the initial value problem, which are proved by the method of perturbation of linear semigroups, a detailed review of other mathematical results in this area is given.

The fourth chapter deals with the analysis of kinetic equations for dense gases with particular attention to the Enskog equation. The first aim is to provide mathematicians with sufficient information on this class of equations, which have certainly a relevant physical interest and are derived under physical assumptions more accurate than the ones of the Boltzmann equation. Then two mathematical problems are studied: the first one refers to the analysis of the initial value problem, the second is the analysis of the equivalence and distance between the solutions of the initial value problem obtained by the Boltzmann and Enskog equations.

The last chapter is a concise presentation of exercises and a brief outline of open research problems.

The book is written for mathematicians who work on nonlinear problems in the kinetic theory of gases and related fields. The book, however, should also be readable to postgraduate students who hope to enter into this field, and to theoretical physicists who want to be aware of the mathematical results available in this field and of the inner pathologies of the Boltzmann and Enskog equations with particular attention to nonlinear effects.

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