POTTS MODELS and RELATED PROBLEMS in STATISTICAL MECHANICS

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Preface

This book is largely a rearrangement of a set of lectures I gave at RIMS, Kyoto in spring 1989. The additional material is either from lectures I *attended* in that period, or arose in the process of rearrangement. The book itself was written at Birmingham University Mathematics Department (summer 89) and City University Mathematics Department (autumn 89 onward).

The central theme of Potts models was chosen with the aim of drawing together, in a systematic way, some key ideas from the continuing explosion of statistical mechanical research. The acid test of relevance to this well established physical picture is one sure way to force a coherent direction on the general explosion of results.

The potential drawback of writing a book on such a vital subject is that one feels more and more pressure each week to incorporate the very latest results. Succumbing to this pressure would obviously result in a rather slowly converging process. In any case, most up to the minute research turns out not, on mature reflection, to be suitable for a textbook. I therefore introduced a cut-off on new material after spring 1989 (except where directly pertinent to the presentation of material already included). At time of going into print no aspect of the subject which has matured since then seems, by its omission, to leave a gap in the presentation.

Of course the subject will soon move on, but the fundamentals will still be fundamental. With this in mind, another aim was to provide a handy work of reference for my own research use, just as Baxter's (1982) book (with a different slant and broader but less recent perspective) has become for so many.

I have tried to make the mathematical development as explicit as is consistent with a finite number of pages. As a physicist I have in particular strived to convey some of the pleasure I have found in learning and applying algebraic techniques to physical problems... and physics is forever trying to teach us algebra.

London, England

Paul Martin

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