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Asymptotic Behavior of Generalized Functions

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Asymptotic Behavior of Generalized Functions

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Preface

There are several approaches to generalized asymptotics within spaces of generalized functions. Probably the most developed approaches are that of Vladimirov, Drozhinov and Zavyalov [192], and of Kanwal and Estrada [56]. The first approach is followed by the authors of this book and extended in the direction of the S-asymptotics. The second approach is related to moment asymptotic expansions of generalized functions and the Cesàro behavior, and we refer to papers of Estrada, Kanwal and Vindas given in the references. We refer to [135] for the different definitions of the asymptotic behavior of distributions.

Asymptotic analysis is a very old and wide branch of mathematics. In general, the state of a system after certain period of time, or the behavior of a system in the frame of some limit procedure described by a suitable mathematical model, leads to the use and development of the methods of asymptotic analysis. With the general development of various branches of mathematical analysis and, especially, the theory of differential equations, the asymptotic analysis has obtained new impulses that resulted in new approaches and methods. In our analysis, such impulses came from the use of slowly varying functions and the introduction of asymptotic behavior of generalized functions.

In the first years of the twentieth century, the needs for a class of functions with growth order between the constant functions and power functions, in the scale of functions used in asymptotic analysis, appeared in many papers. Already Landau [87] used a term similar to “regular variation”, but the theory of regularly varying and slowly varying functions (cf. **0.3**) belongs to Karamata and his pupils. Karamata’s basic results were published in [76] and elaborated in [77]. Although the theory of regularly

varying functions was invented for proving Tauberian type theorems for the Laplace-Stieltjes transform, such functions have been successfully employed in many other areas of mathematics: for Mercerian type theorems, in analytic number theory, the theory of entire functions and in the analysis of various classes of differential equations. Also, Karamata regularly varying functions were recognized as an important tool in probability theory [60].

At the present time the class of regularly varying functions is an integral part of asymptotic analysis. In the monographs [148], [64] and [9] one can find collected together the theory and applications of regular varying functions, the last one being certainly the most comprehensive treatise.

The class of regularly varying functions became more interesting and more applicable after the introduction of the notions of asymptotic behavior and expansion of generalized functions. We refer to [135] for a general overview of the development of the theory of asymptotic behavior of generalized functions up to 1989.

The notion of asymptotic behavior of generalized functions have had a very important role in quantum physics, [8], [10], [191], [192], where rigorous proofs of results in the foundation of this theory were obtained by the use of generalized asymptotic behavior. These papers motivated mathematicians to develop further the methods of asymptotic analysis on spaces of generalized functions.

The first definition of an asymptotic behavior for a generalized function is attributed to Lighthill [91]. His definition can be applied only to the so-called semi-regular distributions. Such a distribution T is equal to a locally integrable function f in a neighborhood of the observed point (in our case, infinity). The asymptotics of the function f at infinity gave the asymptotics of the distribution T at infinity. The same idea was adopted by Jones [72], Lavoine [88], Mangad [95], and Zemanian[203].

The next step forward was made by Silva in his axiomatic approach to distribution theory [147] and later by Lavoine and Misra [89], [90]. Silva introduced the growth order symbols small o and big O , in order to measure the asymptotic behavior of a distribution. Let $r \in C^\infty(a, \infty)$, $a \in \mathbf{R}$ and let $F \in C(a, \infty)$. If a distribution T satisfies

$$T = r \cdot D^m F \text{ in } (a, \infty), \quad m \in \mathbf{N}_0, \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{F(x)}{x^m} = 0,$$

then $T = o(r)$, $x \rightarrow \infty$. If there exists a constant M such that

$$\left| \frac{F(x)}{x^m} \right| \leq M, \quad x \in (b, \infty), \quad b \geq a,$$

then $T = O(r)$, $x \rightarrow \infty$. Finally, if $T = D^m F$ in (a, ∞) and

$$\lim_{x \rightarrow \infty} \frac{F(x)}{x^m} = \frac{C}{m!},$$

then T behaves as the constant C in $\mathcal{D}'(\mathbf{R})$.

Lavoine and Misra defined the notion of “*equivalence at infinity*”. The distribution T is equivalent at infinity with Ax^α , $A \neq 0$, $\alpha \notin -\mathbf{N}$, if there exist $m \in \mathbf{N}_0$, $m + \alpha > 0$, and $F \in C(\mathbf{R})$, such that $T = D^m F$ on (a, ∞) and

$$F(x) \sim AC_{\alpha, m} x^{\alpha+m}, \quad x \rightarrow \infty,$$

$$C_{\alpha, m} = \begin{cases} 1, & m = 0 \\ \Gamma(\alpha + 1)/\Gamma(\alpha + m + 1), & m > 0. \end{cases}$$

Later on, they replaced Ax^a by $Ax^a \log x$ in their definition of equivalence at infinity.

A natural generalization came with the introduction of regularly varying functions (cf. **0.3**) as the basic comparison functions for the asymptotics of generalized functions.

The notion of the quasi-asymptotic behavior (the quasi-asymptotics) of tempered distributions appeared as a qualitatively new important step in the asymptotic analysis of generalized functions. The first paper with the definition of the quasi-asymptotic behavior was the one of Zavyalov [200]. He introduced the quasi-asymptotics as automodel asymptotics “*avtomodel'naya asimptotika*”. First, he defined a class M_+ of tempered distributions as follows: $T \in M_+$ if and only if there exists $a \in \mathbf{R}$ such that $T \in C^\infty(-\infty, a)$ and decreases faster than $|x|^r$, as $x \rightarrow -\infty$, for any positive r . For Zavyalov, $T \in M_+$ has the quasi-limit ($q - \lim$) equal to C if

$$T(kx) \rightarrow CH(x), \quad k \rightarrow \infty \text{ in } \mathcal{S}'(\mathbf{R}),$$

where $H(x)$ is the Heaviside function. In short, $q - \lim_{x \rightarrow \infty} T(x) = C$. If $T \in M_+$ and $q - \lim T^{(\alpha)}(x) = C \neq 0$, $\alpha \in \mathbf{R}$, then T admits a quasi-asymptotic part of degree α at infinity. Zavyalov proved an equivalent

statement: $T \in M_+$ admits a quasi-asymptotic part of degree α at infinity if and only if

$$k^{-\alpha}T(kx) \rightarrow g(x), \quad x \rightarrow \infty \text{ in } \mathcal{S}'(\mathbf{R})$$

for some distribution $g \neq 0$. This second statement has been used thereafter as the definition of the quasi-asymptotics.

Let us emphasize that the predominant role in the development of the quasi-asymptotics belongs to the Russian school with Vladimirov and his pupils and collaborators. Their results have given important impulses to the study of asymptotic behavior in different spaces of generalized functions and their applications. Especially, the robust work of Drozhzhinov and Zavyalov [38], [39], [40], [42], in the last ten years has been an extremely important contribution to the subject.

The notion of the shift asymptotics appeared for the first time in [146], Chapter VII, the second remark after Theorem VI:

$$\inf\{k \in \mathbf{R}; \{T(x+h)/(1+|h|^2)^{k/2}\}_{h \in \mathbf{R}^n} \text{ is a bounded subset of } \mathcal{D}'(\mathbf{R}^n)\}$$

was called the degree of growth of the distribution T at infinity.

The shift asymptotics, defined in [15], has been later called S-asymptotics by the first two named authors of this book. It is said that $T \in \mathcal{S}'_+$ has shift asymptotics at infinity related to the regularly varying function $\rho(h) = h^\alpha L(h)$ if

$$T(x+h)/\rho(h) \rightarrow g(x), \quad h \rightarrow \infty \text{ in } \mathcal{S}'.$$

It was proved that if $\alpha > -1$, then T has quasi-asymptotics related to ρ at infinity. This result can give the impression that the S-asymptotics is a particular case of the quasi-asymptotics; however, this is not really the case. The relation between the S-asymptotics and the quasi-asymptotics is described in this book, Chapter I **2.13**. The first two authors of the book adapted and made precise the S-asymptotic behavior of generalized functions, and studied this type of asymptotic behavior for distributions ultradistributions and hyperfunctions, giving structural characterizations of comparison functions and the limit generalized function. It is worth to mention that the quasi-asymptotics is a natural notion in the analysis of tempered generalized functions, while the framework for the S-asymptotics is the space of exponential distributions $\mathcal{K}'_1$.

It should be mentioned that this book has a minor part in common with the previous books [135] and [192], which are also devoted to the

asymptotic behavior of distributions. This was necessary in order to make the exposition complete and more transparent. Conceptually, we first treat the asymptotic behavior in a dual space $\mathcal{F}'_g$ of a suitable, but rather general, barrelled locally convex space of functions $\mathcal{F}_g$. Many important spaces of generalized functions are of this kind. The study is then specialized to a space $\mathcal{F}'$, a space of distribution, ultradistribution, or Fourier hyperfunction type. In this way we develop a general theory of asymptotics in $\mathcal{F}'$. We then specify the space $\mathcal{F}'$ in order to analyze special properties of asymptotic behaviors or expansions which are intrinsically connected with the nature of each space under consideration. We illustrate the theory by examples and point out some open problems which may serve as an encouragement for further investigations.

The book consists of two parts. The first one deals with the basic properties and analysis of the S-asymptotics and quasi-asymptotics in the space $\mathcal{F}'_g$ and in various special spaces of generalized functions. The asymptotic expansions and the Taylor expansion occupy also an important place in this part of the book.

In the second part one can find several applications of the asymptotic behavior of generalized functions. Abelian and Tauberian type theorems for integral transforms of convolution and Mellin-type convolution are studied for general kernels as well as for various special kernels such as the Laplace, Stieltjes, Weierstrass and Poisson transforms. This part also contains the asymptotic analysis of solutions to linear differential and partial differential equations, and to equations with ultradifferential or local operators. In addition, we give some applications of the quasi-asymptotic behavior of distributions to the study of summability of Fourier series and integrals.

We point out that our exposition, based on the work of the first two authors and their coauthors, has been extended with recent results of the third author and his collaborators. In this way, this book contains novel and strong results which give answers to some old problems in the area. Those results are presented in Chapter I **2.10–2.12** and Chapter II **5**, and make the presentation up-to-date.

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