P-ADIC ANALYSIS AND MATHEMATICAL PHYSICS

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P-ADIC ANALYSIS AND MATHEMATICAL PHYSICS

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INTRODUCTION

Since the Newton and Leibnitz time differential equations over the real number field have been used in mathematical physics. It is not customary to discuss why exactly real numbers should be used. Why has this happened? The point is that physical processes take place in space and time and space-time coordinates are usually considered as real numbers. Since the Euclid time the three-dimensional Euclidean space $\mathbb{R}^3$ has been treated as the physical space. As it is known an important development of this point of view has been done by Riemann and Einstein using the Riemannian geometry, but basically up to now $\mathbb{R}^3$ is a mathematical model for space and $\mathbb{R}^4$ is a model for space-time.

These ideas have become so common, that $\mathbb{R}^3$ is perceived as the true physical space. But in fact *the Euclidean space $\mathbb{R}^3$ is not more than a mathematical model for the real physical space.* In order to convince ourselves that the Euclidean space is a good model for the physical space geometrical axioms from elementary geometry should be checked in practice. To this end lengths of segments, angles etc. should be measured precisely. However in quantum gravity and in string theory it was proved that there is the following obstacle to perform such measurements. If Δx is an uncertainty in a length measurement, then the inequality

$$\Delta x \geq l_p = \sqrt{\frac{hG}{c^3}} \quad (1)$$

takes place. This inequality is stronger than the Heisenberg uncertainty principle. Here h is the Planck constant, c is the velocity of light and G is the gravitational constant. l_p is called the Planck length and it is an extremely small quantity, approximately 10^{-33} cm. We are not going to discuss here a derivation of the fundamental inequality (1) which has a long history. Let us emphasize that by virtue of (1) *a measurement of distances smaller than the Planck length is impossible.*

Let us turn our attention to axioms of Euclidean geometry. In a list of axioms there exists the so-called Archimedean axiom, which was at first pointed out and analyzed by Veroneze and Hilbert. According to the Archimedean axiom any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. Really, this is a physical axiom which concern the process of measurement. Two different scales are compared in this axiom. It means that we can measure distances as small as we want. But as we just discussed, the Planck length is the smallest possible distance that can in principle be measured. So a suggestion emerges to abandon the Archimedean axiom at very small distances. This leads to a non-Euclidean and non-Riemannian geometry of space at small distances.

How can one construct a physical theory corresponding to a non-Archimedean geometry? As it is well known there is an analytical description of geometry. One uses coordinates to describe a geometrical picture.

There are two equivalent approaches

$$\text{geometry} \longleftrightarrow \text{number system.}$$

The usual Euclidean geometry is described by means of real numbers. If we want to abandon the standard geometry for description of small distance in physical space-time we have to abandon real numbers. What should be used instead of real numbers?

In computations in everyday life, in scientific experiments and on computers we are dealing with integers and fractions, that is with rational numbers and we never have dealings with irrational numbers - infinite non-periodic decimals. Results of any practical action we can express only in terms of rational numbers which are considered to have been given us by God. Certainly, there exists generally accepted confidence that if we carry out measurements more and more precisely, then in principle we can get any large number of decimal digits and interpret a result as a real number. However this is an idealization and as it follows from the previous discussion we should be careful with such statements. Thus, *let us take as our starting point the field Q of rational numbers*. A geometric notion of distance corresponds to a notion of a norm on Q . Norm is a real valued function $|x|$ with the following properties

$$1) |x| \geq 0, |x| = 0 \iff x = 0,$$

$$2) |xy| = |x||y|,$$

$$3) |x + y| \leq |x| + |y|$$

for any rational numbers x, y . What norms do exist on $\mathbb{Q}$? There is a remarkable Ostrowski theorem describing all norms on $\mathbb{Q}$. According to this theorem any nontrivial norm on $\mathbb{Q}$ is equivalent to either ordinary absolute value or p -adic norm for some prime number p . p -Adic norm $|x|_p$ is defined by the following. Let us fix a prime number $p = 2, 3, 5, \dots$. Any rational number x can be represented in the form $x = p^\nu \frac{m}{n}$, where ν is an integer and m and n are integers which are not divisible by p . This representation is unique. Then by definition

$$|x|_p = 1/p^\nu. \quad (2)$$

At first sight this definition looks artificial but according to the Ostrowski theorem there are no others nonequivalent norms on $\mathbb{Q}$. It is easy to verify the validity of the properties 1)–3) for $|x|_p$. In fact instead of triangle inequality a stronger inequality takes place

$$3') |x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

A norm that satisfies 3') is called *non-Archimedean*.

The completion of the field $\mathbb{Q}$ of rational numbers with respect to usual absolute value leads to the field $\mathbb{R}$ of real numbers. Analogously, the completion of the field $\mathbb{Q}$ with respect to p -adic norm $|x|_p$ leads to the field $\mathbb{Q}_p$ of p -adic numbers for any prime p .

Any real number can be represented as a decimal, that is a series

$$\pm 10^\nu \sum_{0 \leq n < \infty} b_n 10^{-n}. \quad (3)$$

Analogously, any p -adic number can be represented as a series

$$x = p^\nu \sum_{0 \leq n < \infty} a_n p^n, \quad (4)$$

where a_n are integers, $0 \leq a_n \leq p-1$. If $a_0 \neq 0$, then the representation (4) is unique. Note that in contrast to (3) where we have a power series in the small parameter $1/10$, in (4) we have a power series in the prime number p . The series (4) converges in p -adic norm.

The expression (4) can be considered as a definition of p -adic number. Addition, multiplication etc. of p -adic numbers of the form (4) are carried out as for power series. Any rational number including negative numbers, can be represented in the form (4). For example,

$$-1 = (p+1) + (p-1)p + (p-1)p^2 + \dots$$

p -Adic numbers were introduced by K.Hensel at the end of XIX century. They form an integral part of number theory, algebraic geometry, representation theory and other branches of modern mathematics.

Geometry of the space $\mathbb{Q}_p$ is surprisingly unlike the geometry of the space $\mathbb{R}$, that follows from the property 3') of the norm. In particular the Archimedean axiom is not true in $\mathbb{Q}_p$ ($\mathbb{Q}_p$ is a non-Archimedean space). For two different balls in non-Archimedean geometry one has that either one of them is contained in another or they have no common points. It resembles the behavior of two drops of mercury. *The field of p -adic numbers has a natural hierarchical structure*: every disc consists of the finite number of disjoint discs of smaller radius; the field $\mathbb{Q}_p$ is homeomorphic to a Cantor set on the real line.

Thus we come to the following picture. Fundamental physical theory should be based on rational numbers, at very small distances an important role have to play p -adic numbers and at large distances — real numbers. It leads us to the necessity of reconsidering of the basis of mathematical and theoretical physics beginning from the classical mechanics and right up to the string theory using number theory, p -adic analysis and algebraic geometry. It is a huge volume of material and in this book we cannot involve all of its aspects. The purpose of this book is to give an introduction to those parts of p -adic analysis which are proved to be the most important in applications and to describe some results of p -adic mathematical physics.

There is a natural hierarchical and fractal-like structure in the field of p -adic numbers. Therefore *p -adic analysis and non-Archimedean geometry can be used not only for the description of geometry at small distances, but also for describing chaotic behavior of complicated systems such as spin glasses and fractals in the framework of traditional theoretical and mathematical physics.*

We are only at the beginning of the p -adic mathematical physics. We hope that p -adic numbers would find applications in such fields as the theory of turbulence, biology, dynamical systems, computers, problems of an information transference, cryptography, economy etc. and also in other natural sciences which study systems with chaotic fractal behavior and hierarchical structures.

The book starts from an account of p -adic analysis. Mainly two types of analysis are considered. One is for p -adic valued functions of the type $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ and the other is that of complex valued functions, $f : \mathbb{Q}_p \rightarrow \mathbb{C}$.

The analysis of functions from $\mathbb{Q}_p$ to $\mathbb{Q}_p$ is constructed in many respects

by analogy to ordinary real analysis. For example, the exponential function is defined by the formula

$$e^x = \sum_{0 \leq n < \infty} \frac{x^n}{n!},$$

where $x \in \mathbb{Q}_p$ and the series converges in p -adic norm for $|x|_p < 1$ if $p \neq 2$. Derivatives are defined in a standard manner, but with the help of p -adic norm. They have standard properties. One can write out ordinary and partial differential equations and study properties of their solutions. In particular equations of p -adic classical mechanics for harmonic oscillator have the form

$$\dot{X} = \frac{\partial H}{\partial P}, \quad \dot{P} = -\frac{\partial H}{\partial X},$$

where the Hamiltonian is

$$H = \frac{P^2}{2} + \frac{X^2}{2}. \quad (5)$$

Here time t , coordinate $X = X(t)$ and momentum $P = P(t)$ are p -adic variables. For p -adic valued functions there is no natural analogy of the Lebesgue integral.

On the contrary for functions $f: \mathbb{Q}_p \rightarrow \mathbb{C}$ there exists a natural integral calculus, because on $\mathbb{Q}_p$ as on any locally-compact group there exists the Haar measure. But problems of differentiation of such functions appeared.

Complex valued exponential function (additive character) is defined by the expression

$$\chi_p(x) = \exp 2\pi i \{x\},$$

where $\{x\}$ is the rational part of the series (4). Using the character $\chi_p(x)$ and the Haar measure dx one constructs the Fourier-transform for complex valued functions $f(x)$,

$$\tilde{f}(\xi) = \int_{\mathbb{Q}_p} f(x) \chi_p(\xi x) dx.$$

A complex valued function $f(x)$ cannot be differentiated on p -adic argument x in the usual sense. In the theory of generalized functions over $\mathbb{Q}_p$

the role of differentiability of test functions is played by a condition of local constancy of $f(x)$. An analogy of the operator of differentiation is a pseudodifferential operator

$$Df(x) = \int_{\mathbb{Q}_p} |\xi|_p \tilde{f}(\xi) \chi(-\xi x) d\xi = \frac{p^2}{p+1} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x-y|_p^2} dy. \quad (6)$$

How to compare results of p -adic theory with a theory over real numbers? Let us remind, that only rational numbers are directly observed. Consequently, one should look into rational points of solutions of our equations. For example, for harmonic oscillator (5) let us consider rational solutions of the trajectory in phase space

$$P^2 + X^2 = 1. \quad (7)$$

As it is known, rational solutions of the equation (7) are given by the formulas:

$$P = \frac{m^2 - n^2}{m^2 + n^2}, \quad X = \frac{2mn}{m^2 + n^2},$$

where m and n are integers. Thus, we have an infinite set of rational points on the algebraic curve (7) and the form of the phase trajectory of p -adic harmonic oscillator cannot be experimentally distinguished from the phase trajectory of harmonic oscillator over real numbers. Of course, for other systems the result will be different as the number of rational points on an algebraic curve essentially depends on the type of the curve.

Quantum mechanics is defined by a triple

$$\{L^2(\mathbb{Q}_p), W(z), U(t)\}, \quad (8)$$

where $L^2(\mathbb{Q}_p)$ is the Hilbert space of complex valued functions on $\mathbb{Q}_p$, $W(z)$ is a unitary representation of the Heisenberg-Weyl group in $L^2(\mathbb{Q}_p)$, that is the representation of canonical commutation relation and $U(t)$ is a unitary evolution operator. For harmonic oscillator $U(t)$ is a restriction of unitary representation of metaplectic group on an abelian subgroup. In the book we study the spectral properties of the evolution operator. To this end we investigate an expansion of the representation $U(t)$ over irreducible representations.

How p -adic quantum mechanics correlates with ordinary one? The following formula takes place:

$$\exp[2\pi i(k^2 t - kx)] = \prod_p \chi_p(k^2 t - kx), \quad (9)$$

for rational numbers k, t and x . Thus the wave-function of free particle, the so-called plane wave can be represented as a product of plane waves of p -adic particles. It can be interpreted by saying that an *ordinary free particle consists of p -adic ones*, like an elementary particle consists of quarks.

Formulas like (9) are called Euler or adelic products. A well known example of such formulas is the Euler representation for the zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

An analogous representation takes place for propagator in the field theory:

$$\frac{1}{|k^2|} \prod_p \frac{1}{|k^2|_p} = 1, \quad k \in \mathbb{Q},$$

and also in p -adic string theory.

Elaboration of the formalism of mathematical physics over p -adic number field is an interesting enterprise apart from possible applications, as it promotes deeper understanding of the formalism of standard mathematical physics. One can think that there is the following principle. *Fundamental physical laws should admit a formulation invariant under a choice of a number field*. Thus we include into consideration not only rational, real and p -adic number fields but also other fields.

There are a number of other formulations of quantum mechanics and field theory which are equivalent over the real number field, but they are different over $\mathbb{Q}_p$.

In Euclidean formulation of p -adic quantum mechanics one uses an action

$$S = \int_{\mathbb{Q}_p} [\varphi(x) D^2 \varphi(x) + m^2 \varphi^2(x)] dx. \quad (10)$$

Here φ is a real valued function of p -adic argument and the operator D is defined in (6). It is not clear at the moment in which sense one has

an equivalence of the quantum mechanics (8) and the quantum mechanics with the action (10). One of the possible actions for p -adic field theory has the form

$$S = \int_{\mathbb{Q}_p^n} \tilde{\varphi}(k) a(k) \tilde{\varphi}(-k) dk + \int_{\mathbb{Q}_p^n} V(\varphi(x)) dx$$

where $a(k)$ is a function on k and $V(\varphi)$ describes an interaction. For this action the quantum field theory is developed using ordinary methods of theoretical physics, such as classical equations, path integrals etc.

The modern string theory began as it is known from the Veneziano amplitude describing scattering of four particles

$$A(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (11)$$

Here a and b are parameters depending on momenta of colliding particles. The function x^a is a multiplicative character on the real line. One can interpret (11) as a convolution of two characters. Then a generalization of the Veneziano amplitude will be of the following expression

$$A(\gamma_a, \gamma_b) = \int_K \gamma_a(x) \gamma_b(1-x) dx, \quad (12)$$

where K is a field, γ_a is a character and dx is a measure on K . For different K , in particular for K equals to $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ or the Galois field $\mathbb{F}_p$ one gets formulas important in string theory and number theory. The Veneziano amplitude is a special case of more general Koba-Nielsen amplitude which in abstract form looks like

$$A_n = \int \prod_{i < j} \gamma_{a_{ij}}(x_i - x_j) dx_1 \dots dx_n.$$

If we take $K = \mathbb{Q}_p$ we come to a theory of p -adic dual amplitudes and strings. In the development of p -adic mathematical physics the adelic formula

$$\prod_{p=2}^{\infty} \int_{\mathbb{Q}_p} |x|_p^a |1-x|_p^b dx = 1$$

discovered by Freund and Witten is important.

It is possible to extend the formula (11) in another way using the p -adic valued gamma function. In this way there appeared interesting connections of p -adic strings with the Jacobi sums and l -adic cohomology of the Fermat curves over the Galois fields .

There exists an unexpected connection of p -adic analysis with q -analysis and quantum groups and thus with noncommutative geometry. q -analysis is sort of a q -deformation of ordinary analysis, it was spared by such mathematicians as Euler, Gauss and others. Spherical functions on quantum groups are q -special functions. The Haar measure on the quantum group $SU_q(2)$ can be expressed in terms of q -integral

$$\int_0^1 f(x) d_q x = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n . \quad (13)$$

Here $0 < q < 1$ and $f(x)$ is a real function. On the other hand the integral with respect to the Haar measure on $\mathbb{Q}_p$ has the form

$$\int_{|x|_p \leq 1} f(|x|_p) dx = \left(1 - \frac{1}{p}\right) \sum_{n=0}^{\infty} f(p^{-n}) p^{-n} . \quad (14)$$

We see that for $q = \frac{1}{p}$ the expressions (13) and (14) are equal. There exists a number of other surprising relations between q -deformations and p -adic analysis.

Let us summarize in conclusion general directions of applications of p -adic analysis in mathematical physics

- geometry of space-time at small distances
- extensions of the formalism of theoretical physics to other number fields
- classical and quantum chaos; investigation of complicated systems such as spin glasses and fractals
- adelic formulas giving a decomposition of certain physical systems on more simple parts
- stochastic processes and probability theory
- connections with q -analysis and quantum groups.

In concluding the Introduction let us give a short guide for reader. First of all we would like to emphasize that no knowledge of p -adic analysis is assumed. It is not necessary to read this book section by section. The

interdependence table is very simple. After reading the first two subsections (1.1) and (1.2) containing the definitions of p -adic norms and p -adic numbers, one can read any other section.

* * *

We are very grateful to all our friends and colleagues for the encouragement they gave and numerous enthusiastic discussions of exciting topics covered in this book.

P-ADIC ANALYSIS AND MATHEMATICAL PHYSICS