An Invitation to q -Series

From Jacobi's Triple Product Identity to
Ramanujan's "Most Beautiful Identity"

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From Jacobi's Triple Product Identity to
Ramanujan's "Most Beautiful Identity"

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Preface

Around ten years ago, I came across the following result of Srinivasa Ramanujan:

$$\frac{e^{-2\pi/5}}{1 + \frac{e^{-2\pi}}{1 + \frac{q^{-4\pi}}{1 + \frac{q^{-6\pi}}{1 + \ddots}}}} = \sqrt{2 + \beta} - \beta, \quad (0.1)$$

where $\beta = (1 + \sqrt{5})/2$ is the Golden Ratio. I said to myself, “Wow? How did Ramanujan figure this out?” Indeed, Hardy (1999, p. 9) said Eq. (0.1) was one of the three identities which defeated him completely. Since then I have started to write some notes on the subject. Later these notes became some of my teaching materials. Eventually, they grew to become this book.

This book is still growing: I will maintain a link at my website at

<https://edocs.uis.edu/hchan1/www/>

listing all the errors and typos. Feel free to email me your comments (for my current email, see my website). If time permits, I will add some additional materials related to the book on my website.

By the way, if you are interested to know how Eq. (0.1) can be proven but have never taken a course in q -series, you can read Chapters 2, 7, 11, and 12.

I would like to express my deepest gratitude to many people, without whom this book will never see the light of day.

I want to thank See Tsai, my wife and soul mate. Without her love, support and prayers (and those of my mom too!), I don’t think the completion of this book project would be possible. Thanks also to our son Lemuel

for bringing us so much joy in our life. Lemuel is five-year old and does not know the existence of the book project. But I am sure he felt some of his play time with daddy was somehow cut short. Great thanks to my parents, Fan and Kim Chan, for their love and support.

I want to thanks my colleagues at the math department at UIS, in particular John Snyder, Patty Stoutamyer, and Prof. Chung-Hsien Sung for their tremendous support. The reduced teaching load in Spring 2010, granted to me by the College of Liberal Arts and Sciences, could not come at a better time and I am truly grateful for that. Thanks to all the students who listened to various parts of the materials in this book, in particular, Dan Collins, Scott Ebbing, Cassie Koehne, Allison Leight, Mike Roberts and Tiffany Whitlow.

I want to thank many mathematicians, from whom I have learned so much. In particular, I want to thank Professors G. E. Andrews, B. C. Berndt and M. D. Hirschhorn for their inspiring work, and they are always very encouraging.

I want to thank Rok Ting Tan for being such a wonderful editor, and Prof. K. K. Phua for his invitation to write this book. I received his invitation on August 8, 2008, which was also the opening day of the 2008 Summer Olympics in Beijing. I will not forget that day.

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Contents

<i>Preface</i>	v
1. Introduction	1
Part I: Jacobi's Triple Product Identity	3
2. First proof (via functional equation)	5
3. Second proof (via Gaussian polynomials and the q -binomial theorem)	11
4. Some applications	21
5. The Boson-Fermion correspondence	31
6. Macdonald's identities	37
Part II: The Rogers-Ramanujan Identities	45
7. First proof (via functional equation)	47
8. Second proof (involving Gaussian polynomials and difference equations)	53
9. Third proof (via Bailey's lemma)	69

10. Excursus: Mock theta functions	85
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Part III: The Rogers-Ramanujan Continued Fraction **95**

11. A list of theorems to be proven	97
12. The evaluation of the Rogers-Ramanujan continued fraction	105
13. A “difficult and deep” identity	119
14. A remarkable identity from the Lost Notebook and cranks	125
15. A differential equation for the Rogers-Ramanujan continued fraction	137

Part IV: From the “Most Beautiful Identity” to Ramanujan’s congruences **145**

16. Proofs of the “Most Beautiful Identity”	147
17. Ramanujan’s congruences I: analytical methods	157
18. Ramanujan’s congruences II: an introduction to t -cores	169
19. Ramanujan’s congruences III: more congruences	185
20. Excursus: modular forms and more congruences for the partition function	195
Appendix A Proofs of $\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$	205
Appendix B The ranks of the partitions of $n = 9$	209
Appendix C The cranks of the partitions of $n = 9$	211