FRACTIONAL CALCULUS

AN INTRODUCTION FOR PHYSICISTS

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Foreword

Theoretical physics has evolved into a multi-faceted science. The sheer amount of knowledge accumulated over centuries forces a lecturer to focus on the mere presentation of derived results. The presentation of strategies or the long path to find appropriate tools and methods often has been neglected.

As a consequence, from a student's point of view, theoretical physics seems to be a construct of axiomatic completeness, where apparently is no space left for speculations and new approaches. The limitations of presented concepts and strategies are not obvious until these tools are applied to new problems.

Especially the creative process of research, which evolves beyond the state of mere reception of already known interrelations is a particular ambition of most students.

One prerequisite to reach that goal is a permanent readiness, to question even well-established results and check the validity of common statements time and time again. Already Roger Bacon in his *opus majus* pointed out, that the main causes of error are the belief in false authority, the force of habit, the ignorance of others and pretended knowledge [Bacon(1267)].

New concepts and new methods depend on each other. The unified theory of electromagnetism by James Clerk Maxwell was motivated by Faraday's experiments. The definite formulation was then realized in terms of a new theory of partial differential equations. Einstein's general relativity is based on experiments on the constancy of the speed of light; the elegant presentation was made possible with Riemann's tensor calculus.

This book too takes up an idea which is simple at first: due to the fact that differential equations play such a central role in physics since the times of Newton and since the first and second derivatives denote such

fundamental properties as velocity and acceleration, does it make sense, to investigate the physical meaning of a half, a π -th and an imaginary derivative?

This interesting question has been raised since the days of Leibniz and has been discussed by all mathematicians of their times. But for centuries no practical applications could be imagined.

During the last years fractional calculus has developed rapidly. This process is still going on, but we can already recognize, that within the framework of fractional calculus new concepts and strategies emerge, which make it possible, to obtain new challenging insights and surprising correlations between different branches of physics.

This is the basic purpose of this book: to present a concise introduction to the basic methods and strategies in fractional calculus and to enable the reader to catch up with the state of the art on this field and to participate and contribute in the development of this exciting research area.

In contrast to other monographs on this subject, which mainly deal with the mathematical foundations of fractional calculus, this book is devoted to the application of fractional calculus on physical problems. The fractional concept is applied to subjects in classical mechanics, group theory, quantum mechanics, nuclear physics, hadron spectroscopy up to quantum field theory and will surprise the reader with new intriguing insights.

This book provides a skillful insight into a vividly growing research area and opens the reader's mind to a world yet completely unexplored. It encourages the reader to participate in the exciting quest for new horizons of knowledge.

Richard Herrmann
Dreieich, Germany
Summer, 2010

Acknowledgments

As a young student, I had the opportunity to attend the Nobel Laureate Meeting at Lindau/Germany in 1982. Especially Paul Dirac's talk was impressive and highly inspiring and it affirmed my aspiration to focus my studies on theoretical physics.

I joined the Institute for Theoretical Physics in Frankfurt/Germany. Working on the fission properties of super-heavy elements, I suggested to linearise the collective Schrödinger equation used in nuclear collective models and to introduce a new collective degree of freedom, the collective spin. Within this project, my research activities are concentrated on the field theoretical implications, which follow from linearised wave equations. As a direct consequence the question of the physical interpretation of multifactorized wave equations and of a fractional derivative came up.

I achieved a breakthrough in my research program in 2005: I found that a specific realization of the fractional derivative suffices to describe the properties of a fractional extension of the standard rotation group $SO(n)$. This was the key result to start an investigation of symmetries of fractional wave equations and the concept of a fractional group theory could be successfully realized. Up to now, this concept has led to a vast amount of intriguing and valuable results.

This success would not have been possible without the permanent enduring loving support of my family and friends.

I am grateful to all those who shared their knowledge with me and made suggestions reflected in this book. Special thanks go to Anke Friedrich and Günter Plunien for their encouragement and their valuable contributions.

I also want to emphasize, that fractional calculus is a worldwide activity. I benefited immensely from international support and communication. Discussions and correspondence with Eberhard Engel, Ervin Goldfain, Cesar

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