Differential Equations

An Introduction to Basic Concepts,
Results and Applications

Second Edition

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Results and Applications

Second Edition

Ioan I Vrabie

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To the Memory of my Parents

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Preface to the Second Edition

The Second Edition of the book differs from the First Edition in that several additions and changes were done. First, in order to offer the reader some basic information concerning two important topics which, in one way or another, are related to differential equations, the following insertions were operated. So, keeping in mind that Volterra Equations are not only nontrivial extensions of differential equations but one of the starting points of research in Operator Theory, we have included Chapter 8, entirely devoted to this old but still modern and of great interest topic. Once this being done, taking into consideration that a very efficient tool in solving linear integral equations of Volterra type is the Laplace Transform, we have decided to present the fundamentals of this theory in Section 10.4. Further, we have added Chapter 9 dealing with the Calculus of Variations, a discipline having its roots in Mechanics and which is essentially based on nonlinear differential equations. It is almost needless to say that the Calculus of Variations is indispensable not only for mathematicians but for physicists as well. In addition, this branch of Mathematics was – and still is – the starting point of many deep researches in both Differential Equations and Differential Geometry.

As in the case of all other chapters, each of Chapters 8 and 9 ends with an exercises and problems section. The complete solutions of the exercises and problems newly added were included in the last section of the book.

Second, trying to make the book as self-contained as possible, a simple form of the Implicit Functions Theorem, i.e. Theorem 2.8.1, whose proof was derived from Peano's Theorem 2.2.1, was included. In addition, an extension to the case of implicitly defined vector-valued functions was proposed in Problem 2.28. It should be noticed that the proof here presented, in spite of its relative simplicity – due, of course, to the very deep result

on which is based, i.e. Peano's Theorem 2.2.1 –, is, with no doubt, far from being elementary. However, the reader which is familiar with Peano's Theorem 2.2.1 would have no problem in following the arguments there involved.

It should be emphasized that both newly added chapters and sections are intended as an incentive for further study rather than a main source of information.

Several new entries on the reference list were included. These are [Arnold (1978)], [Barbu (2010)], [Bliss (1946)], [Cârjă *et al.* (2007)], [Gelfand and Fomin (1963)], [Giusti (2003)], [Havârneanu (2007)], [Krasnov *et al.* (1977)], [Lavrentiev and Sabat (1959)], [Lefter (2006a)], [Lefter (2006b)], [Sneddon (1971)] [Şabac (1981)], [Widder (1946)] and two of them, i.e. [Cârjă *et al.* (2004)] and [Cârjă and Vrabie (2005)], which were just in print at the time when the First Edition has appeared, were updated. Some small corrections were operated and a direct proof of the implication (iii) implies (i) in Theorem 4.1.4 was included, substituting the indirect proof given in the First Edition.

Acknowledgements. I gratefully acknowledge the help of Professors Teodor Havârneanu, Cătălin-George Lefter and Mihai Necula, from the Department of Mathematics of “Al. I. Cuza” University of Iaşi, and that of Professor Marius Popescu, from the Department of Mathematics of the University of Galaţi, for the very careful reading of the newly added parts to the Second Edition of the book, and for the useful suggestions and remarks they have made. All their comments, which I took into consideration, have led to a substantial improvement of the presentation. I would also like to mention that Example 9.4.1 is due to Professor Eugen Popa, from the Department of Mathematics of “Al. I. Cuza” University of Iaşi.

I take this opportunity to express my warmest thanks to all of them as well as to the editors of World Scientific for the excellent job of proofreading the whole manuscript.

Iaşi, January 31st, 2011

Ioan I. Vrabie

Preface to the First Edition

The book is an entirely rewritten English version of the lecture notes of a course on Differential Equations I taught during the last twelve years at the Faculty of Mathematics of “Al. I. Cuza” University of Iași. These lecture notes were written in 1999 in Romanian. Their goal was to present in a unitary frame and from a new perspective the main concepts and results belonging to a discipline which, due to the continuous interplay between theory and applications, is by far one of the most fascinating branches of modern mathematics, i.e. *differential equations*. It was my intention to give the reader the opportunity to know a point of view — rather different from the traditional one — offering a possible way to learn differential equations with main emphasis on the Cauchy problem. So, I decided to treat separately the problems of: existence, uniqueness, approximation, continuation of the solutions and, at the same time, to give the simplest possible but complete proofs to some fundamental results which are at the core of the discipline: Peano’s local existence theorem, the classification of non-continuable solutions from the viewpoint of their behavior at the end of the existence interval, the continuous dependence of the solution on the data and parameters, *etc.* This goal was by far very hard to accomplish due to the existence of a long list of very good, or even exceptional, textbooks and monographs on this subject covering all levels of difficulty: [Arnold (1974)], [Arrowsmith and Place (1982)], [Barbu (1985)], [Braun (1983)], [Coddington and Levinson (1955)], [Corduneanu (1977)], [Cronin (1980)], [Elsgolts (1980)], [Halanay (1972)], [Hale (1969)], [Hartman (1964)], [Hirsch and Smale (1974)], [Hubbard and West (1995)], [Ionescu (1972)], [Piccinini *et al.* (1984)], [Pontriaghin (1969)], to cite only a few. However, in spite of this challenging competition, I hope that the reader will find this text attractive enough from both the viewpoint of the chosen topics and

the presentation.

The book contains a preface, a list of symbols, seven main chapters, a short chapter on auxiliary results, a rather long section including detailed solutions to all exercises and problems, a bibliography and ends with an index. With the sole exception of Chapters 6 and 7, which require some basic results on Lebesgue integral and Measure Theory, it is completely accessible to any reader having satisfactory knowledge of Linear Algebra and Mathematical Analysis. The 36 figures included illustrate the concepts introduced and smooth the way towards a complete understanding of the arguments used in the proofs.

The first chapter starts with a very brief presentation of the main steps made along the last four centuries toward the modern theory of differential equations. It continues with some preliminary notions and results referring to: the concept of solution, some methods of solving elementary equations, various mathematical models described either by differential equations or systems of differential equations, and some basic integral inequalities.

The second chapter contains several fundamental results concerning the Cauchy Problem: the local existence, the continuation of the solutions, the existence of global solutions, the relationship between the local and the global uniqueness, the continuous dependence and the differentiability of the solutions with respect to the data and to the parameters.

The third chapter is merely concerned with some classical facts about the approximation of the solutions: the method of power series, the method of successive approximations, the method of polygonal lines, the implicit Euler method and a particular, and therefore simplified, instance of Crandall–Liggett exponential formula.

In the fourth chapter we apply the previously developed theory to a systematic study of one of the most important class of systems: first-order linear differential systems. Here we present the main results concerning the global existence and uniqueness, the structure of the space of solutions, the fundamental matrix and the Wronskian, the variation of constants formula, the properties of the mapping $t \mapsto e^{tA}$ and the basic results referring to n^{th} -order linear differential equations.

The fifth chapter is mainly concerned with the study of an extremely important problem of the discipline: the stability of solutions. We introduce four concepts of stability and we successively study the stability of the null solution of linear systems, perturbed systems and fully nonlinear systems respectively, in the last case by means of the Lyapunov's function method. We also include some facts about instability which is responsible for the

so-called *unpredictability* and *chaos*.

In the sixth chapter, we start with the study of the concept of prime integral, first for autonomous, and thereafter for non-autonomous systems. Next, with this background material at hand, we present the basic results concerning linear and quasi-linear first-order partial differential equations. Some examples of conservation laws are also included.

The seventh chapter, rather heterogeneous, has a very special character being conceived to help the reader to go deeper within this discipline. So, here, we discuss some concepts and results concerning distributions and solutions in the sense of distributions, Carathéodory solutions, differential inclusions, variational inequalities, viability and invariance and gradient systems.

In the last chapter we include some auxiliary concepts and results needed for a good understanding of some parts of the book: the operator norm of a matrix, compact sets in $C([a, b]; \mathbb{R}^n)$, the projection of a point on a convex set.

Each chapter, except the one on Auxiliary Results, ends with a special section containing exercises and problems ranging from extremely simple to challenging ones. The complete proofs of all these are included into a rather developed final section (more than 60 pages).

Acknowledgements. The writing of this book was facilitated by a very careful reading of some parts of the manuscript, by several remarks and suggestions made by Professors Ovidiu Cârjă, Mihai Necula from “Al. I. Cuza” University of Iași, by Professors Silvia–Otilia Corduneanu and Silviu Nistor from “Gh. Asachi” Technical University of Iași, remarks and suggestions which I took into account. The simplified version of the Frobenius theorem was called to my attention by Dr. Constantin Vârsan, Senior Researcher at The Mathematical Institute of the Romanian Academy in Bucharest. Some of the examples in Physics and Chemistry have been reformulated taking into account the remarks made by Professors Dumitru Luca and Gelu Bourceanu. Professor Constantin Onică had a substantial contribution in solving and correcting most part of the exercises and the problems proposed. A special mention deserves the very careful — and thus critical — reading of the English version by Professor Mircea Bârsan.

It is a great pleasure to express my appreciation to all of them.

Iași, November 30th, 2003

Ioan I. Vrabie

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List of Symbols

$\mathcal{A}^\tau$	— the transpose of the matrix $\mathcal{A} \in \mathcal{M}_{n \times n}(\mathbb{R})$
$B(\xi, r)$	— the closed ball centered at ξ of radius $r > 0$
$\overset{\circ}{B}$	— the interior of the set B
$\mathcal{B}^*$	— the adjoint of the matrix $\mathcal{B} \in \mathcal{M}_{n \times n}(\mathbb{R})$
$\text{conv } F$	— the convex hull of F , i.e. the set of all convex combinations of elements in F
$\overline{\text{conv } F}$	— the closed convex hull of F , i.e. the closure of $\text{conv } F$
$\mathcal{CP}(\mathbb{I}, \Omega, f, a, \xi)$	— the Cauchy problem $x' = f(t, x)$, $x(a) = \xi$, where $f : \mathbb{I} \times \Omega \rightarrow \mathbb{R}^n$, $a \in \mathbb{I}$ and $\xi \in \Omega$
$C([a, b]; \mathbb{R}^n)$	— the space of continuous functions from $[a, b]$ to $\mathbb{R}^n$
$\mathcal{D}$	— the data $(\mathbb{I}, \Omega, f, a, \xi)$. So, $\mathcal{CP}(\mathcal{D})$ denotes $\mathcal{CP}(\mathbb{I}, \Omega, f, a, \xi)$
$\text{dist}(\mathcal{K}, \mathcal{F})$	— the distance between the sets $\mathcal{K}$ and $\mathcal{F}$, i.e. $\text{dist}(\mathcal{K}, \mathcal{F}) = \inf\{\ x - y\ ; x \in \mathcal{K}, y \in \mathcal{F}\}$
$\text{dist}(\eta, \mathcal{K})$	— the distance between the point η and the set $\mathcal{K}$, i.e. $\text{dist}(\eta, \mathcal{K}) = \inf\{\ \eta - \xi\ ; \xi \in \mathcal{K}\}$
$\mathcal{D}(\mathbb{R})$	— the space of indefinite differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, with compact support
Δ	— the compact set $[a, a + h] \times B(\xi, r)$
$\delta J[q][\eta]$	— the first variation of J calculated at q along the direction η , i.e. $\frac{d}{d\varepsilon} J[q + \varepsilon\eta] _{\varepsilon=0}$
$\delta^2 J[q][\eta]$	— the second variation of J calculated at q along the direction η , i.e. $\frac{d^2}{d\varepsilon^2} J[q + \varepsilon\eta] _{\varepsilon=0}$
$\mathcal{D}'(\mathbb{R})$	— the set of linear continuous functionals from $\mathcal{D}(\mathbb{R})$ to $\mathbb{R}$
$\text{graph}(x)$	— the graph of $x : \mathbb{I} \rightarrow \mathbb{R}^n$, i.e. $\text{graph}(x) = \{(t, x(t)); t \in \mathbb{I}\}$
$\mathbb{I}, \mathbb{J}, \mathbb{K}$	— nonempty intervals in $\mathbb{R}$
$\mathcal{L}[f(t)](p)$	— the Laplace Transform of the function f
$\mathbb{N}$	— the set of nonnegative integers, i.e. $0, 1, 2, \dots$
$\mathbb{N}^*$	— the set of positive integers, i.e. $1, 2, 3, \dots$
$\nabla_x z$, or ∇z	— the gradient of the function z with respect to $x_1, x_2, \dots, x_n$, i.e. $\nabla_x z = \text{grad}_x z = \left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \dots, \frac{\partial z}{\partial x_n} \right)$
$\ \mathcal{A}\ _{\mathcal{M}}$	— the norm of the matrix $\mathcal{A} \in \mathcal{M}_{n \times m}(\mathbb{R})$, i.e. $\ \mathcal{A}\ _{\mathcal{M}} = \sup\{\ \mathcal{A}x\ _n; x \in \mathbb{R}^m, \ x\ _m \leq 1\}$

Ω	— a nonempty and open subset in $\mathbb{R}^n$
ω_f	— the growth index of the function f , i.e. $\omega_f = \inf\{\omega \in \mathbb{R}; \exists M > 0, f(t) \leq Me^{\omega t}, \forall t > 0\}$
$\mathcal{P}_K(x)$	— the projection of the vector $x \in \mathbb{R}^n$ on the subset $K \subset \mathbb{R}^n$
$\mathbb{R}$	— the set of real numbers
$\mathbb{R}^*$	— the set of real numbers excluding 0
$\mathbb{R}_+$	— the set of nonnegative real numbers
$\mathbb{R}_+^*$	— the set of positive real numbers
$\text{supp } \phi$	— the set $\{t \in \mathbb{R}; \phi(t) \neq 0\}$
Σ	— a nonempty and locally closed subset in Ω
x'	— the derivative of the function x
$\dot{x}$	— the derivative in the sense of distributions of the distribution x
$\mathcal{X}^{-1}(s)$	— the inverse of the matrix $\mathcal{X}(s)$, i.e. $\mathcal{X}^{-1}(s) = [\mathcal{X}(s)]^{-1}$

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