A Second Step to Mathematical Olympiad Problems

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A SECOND STEP TO MATHEMATICAL OLYMPIAD PROBLEMS

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To Lee Peng Yee, a true friend and all-round nice guy, who was
instrumental in finding a publisher for both Steps.

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Foreword

This is the second of two volumes on some higher level problem solving and largely contains material that is not taught in most Western schools. Like the first volume (*A First Step to Mathematical Olympiad Problems*), most of the content here was developed initially as an aid for New Zealand students who were hoping to get to an International Mathematical Olympiad (IMO). This original material was presented in the form of 15 yellow booklets. The first eight of these are the basis of *First Step*; the remaining seven underpin this volume. However, it should be noted that there are also new pieces here. The whole of Chapter 3 was not in the initial yellow booklets neither were many sections and problems that are dotted around the rest of the book.

This book should help improve the background of students preparing for regional and national mathematics competitions by providing them with theorems, ideas and techniques that they may not have already met. It will also give them suggestions on how to approach problems that they have not seen before. But the step to the IMO from regional and national competitions is likely to be a big one and the First and Second Steps should not be considered as a total preparation for an IMO. Hopefully though, they are good stepping stones along the way.

But I hope that the book will also be read by both students and teachers who are interested in mathematics. There are sufficiently many problems here to while away many happy and frustrating hours. I hope that this book encourages many readers to simply enjoy mathematics for its own sake and for the pleasure of getting a problem out.

Finally I would like to thank Irene Goodwin (who is now, unfortunately, no longer with us), Lenette Grant and Leanne Kirk for their help to me while the yellow booklets were being prepared and in the many years since, and to Zhang Ji for her work in turning yellow booklets into Steps.

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