The Chaotic Pendulum

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Moshe Gitterman

Bar-Ilan University, Israel

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Preface

The second part of the title of this book is familiar to everybody who swings to and fro in childhood, or has changed his/her mind back and forth from joy to grief in mature age, whereas the first part of the title needs explanation. We use the word “chaotic” as a synonym for “unpredictable.” Everything was clear until the third quarter of the last century: all phenomena in Nature were either deterministic or chaotic (random). The solution of a second-order differential equation contains two arbitrary constants which can be found from the initial conditions. By defining the coordinate and velocity of the particle at time $t = 0$, one can calculate these variables deterministically at each later time $t > 0$. On the other hand, if a system, say, a Brownian particle, is subject to a random number of collisions with small particles, its motion will be non-predictable (random). However, everything changed in the 1970s, as can be seen from the title of a 1986 conference on “chaos”: “Stochastic Behavior Occurring in a Deterministic System.” Although the idea of “deterministic chaos” appeared earlier, the intensive study of this phenomenon only started about forty years ago. The answer to the question in the title of the article [1], “Order and chaos: are they contradictory or complimentary?”, is now obvious.

The concept of “chaos” is usually associated with systems having a large number of degrees of freedom. The approach of statistical mechanics allows one to calculate the average characteristics of the system, leaving the behavior of individual particles as “random”. It turns out, however, that chaos may appear in a differential equation with only three variables, provided that the system is nonlinear. It is crucial to distinguish between linear and nonlinear differential equations. An important property of chaotic nonlinear equations is the exponential increase in time of their solutions when one makes even the smallest change in the initial conditions. “Deterministic

chaos” appears without any random force in the equations. Such a situation is very common since an infinite number of digits is required to specify the initial conditions precisely, an accuracy that is obviously unattainable in a real experiment. The exponential dependence on initial conditions is popularly known as the “butterfly effect,” which means that an infinitesimal change in initial conditions produces very different results in the long run (the flapping of a butterfly’s wing in Texas may create large changes in the atmosphere leading to a tornado in the Pacific Ocean). A very small change in initial conditions can transform a deterministic system into a chaotic system. Therefore, considering “chaotic motion” means considering the general properties of nonlinear differential equations (Chapter 2), as well the effect of a random force (Chapter 3). The chaotic behavior of a spring, double and spherical pendula are the subject of Chapter 4. A general introduction to the subject can be found in my previous book [2].

Two main features characterize the study of chaos. First, nonlinear dynamics and chaos are an area of intensive mathematical investigation. Second, due to the lack of analytical solutions, this field is usually studied by numerical methods. This book does not contain rigorous mathematical statements or details of numerical methods related to chaos. My aim was to make the presentation as simple as possible, so that a scientist or a student having only a general knowledge of mathematical physics, could easily find in this small volume all the required information for their theoretical or laboratory work.

Contents

<i>Preface</i>	v
<i>List of Equations</i>	xi
1. Pendulum Equations	1
1.1 Mathematical pendulum	1
1.2 Period of oscillations	5
1.3 Underdamped pendulum	10
1.4 Nonlinear vs linear equation	15
1.5 Isomorphic models	16
1.5.1 Brownian motion in a periodic potential	16
1.5.2 Josephson junction	16
1.5.3 Fluxon motion in superconductors	17
1.5.4 Charge density waves	17
1.5.5 Laser gyroscope	18
1.5.6 Synchronization phenomena	18
1.5.7 Parametric resonance in anisotropic systems	18
1.5.8 Phase-locked loop	19
1.5.9 Dynamics of adatom subject to a time-periodic force	19
1.5.10 The Frenkel-Kontorova model (FK)	19
1.5.11 Solitons in optical lattices	20
1.5.12 Other applications	20
1.6 General concepts	20
1.6.1 Phase space	21
1.6.2 Poincare sections and strange attractors	21
1.6.3 Lyapunov exponent	22

1.6.4	Correlation function	22
1.6.5	Spectral analysis	22
1.6.6	Period doubling and intermittency	23
2.	Deterministic Chaos	27
2.1	Damped, periodically driven pendulum	27
2.1.1	Transition to chaos	27
2.1.2	Two external periodic fields	32
2.1.3	Dependence on driving frequency	34
2.1.4	Role of damping	35
2.1.5	Symmetry and chaos	36
2.1.6	Diffusion in a chaotic pendulum	39
2.2	Analytic methods	41
2.2.1	Period-doubling bifurcations	42
2.2.2	Melnikov method	45
2.3	Parametric periodic force	48
2.3.1	Pendulum with vertically oscillating suspension point	49
2.3.2	Transition to chaos	49
2.3.3	Melnikov method	51
2.3.4	Parametric periodic non-harmonic force	52
2.3.5	Downward and upward equilibrium configurations	55
2.3.6	Boundary between locked and running solutions	56
2.3.7	Pendulum with horizontally oscillating suspension point	58
2.3.8	Pendulum with both vertical and horizontal oscillations of the suspension point	62
2.4	Parametrically driven pendulum	62
2.5	Periodic and constant forces	65
2.5.1	Melnikov method	66
2.6	Parametric and constant forces	69
2.6.1	Harmonic balance method	70
2.6.2	Heteroclinic and homoclinic trajectories	71
2.6.3	Numerical calculations	72
2.7	External and parametric periodic forces	73
3.	Pendulum subject to a Random Force	77
3.1	Noise	77
3.1.1	White noise and colored noise	77
3.1.2	Dichotomous noise	78

3.1.3	Langevin and Fokker-Planck equations	79
3.2	External random force	81
3.3	Constant and random forces	82
3.4	External periodic and random forces	85
3.4.1	Two sources of noise	85
3.4.2	Fokker-Planck equation	86
3.4.3	Ratchets	86
3.4.4	Absolute negative mobility	88
3.5	Pendulum with multiplicative noise	89
3.6	Parametric periodic and random forces	91
3.7	Damped pendulum subject to a constant torque, periodic force and noise	92
3.8	Overdamped pendulum	93
3.8.1	Additive white noise	94
3.8.2	Additive dichotomous noise	96
3.8.3	Multiplicative dichotomous noise	99
3.8.4	Additive and multiplicative white noise	102
3.8.5	Multiplicative dichotomous noise and additive white noise	109
3.8.6	Correlated additive noise and multiplicative noise . .	110
4.	Systems with Two Degrees of Freedom	113
4.1	Spring pendulum	113
4.1.1	Dynamic equations	114
4.1.2	Chaotic behavior of a spring pendulum	118
4.1.3	Driven spring pendulum	120
4.2	Double pendulum	123
4.3	Spherical pendulum.	126
5.	Conclusions	131
	<i>Bibliography</i>	133
	<i>Glossary</i>	139
	<i>Index</i>	141

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List of Equations

Underdamped Pendulum

Mathematical pendulum 1

$$\frac{d^2\varphi}{dt^2} + \frac{g}{l} \sin \varphi = 0$$

Damping pendulum 10

$$\frac{d^2\varphi}{dt^2} + \alpha \frac{d\varphi}{dt} + \frac{g}{l} \sin \varphi = 0$$

External constant force 11

$$\frac{d^2\varphi}{dt^2} + \alpha \frac{d\varphi}{dt} + \frac{g}{l} \sin \varphi = a$$

External periodic force 14

$$\frac{d^2\varphi}{dt^2} + \alpha \frac{d\varphi}{dt} + \frac{g}{l} = A \cos(\Omega t)$$

External periodic and constant force 65

$$\frac{d^2\varphi}{dt^2} + \alpha \frac{d\varphi}{dt} + \frac{g}{l} = a + A \cos(\Omega t)$$

Two external periodic forces 33

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi = A_1 \sin(\Omega_1 t) + A_2 \sin(\Omega_2 t)$$

Two external periodic and constant force 33

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi = a + A_1 \sin(\Omega_1 t) + A_2 \sin(\Omega_2 t)$$

External periodic and constant force with quadratic damping 67

$$\frac{d^2\varphi}{dt^2} + \alpha \left(\frac{d\varphi}{dt} \right)^2 + \frac{g}{l} = a + A \cos(\Omega t)$$

Parametric periodic force (vertical oscillations of suspension point) 48

$$\frac{d^2\varphi}{dt^2} + \alpha \frac{d\phi}{dt} + [B + A \sin(\Omega t)] \sin \phi = 0$$

Parametric periodic force (horizontal oscillations of suspension point) 58

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi + A \sin(\Omega t) \cos \varphi = 0$$

Parametric periodic non-harmonic force 53

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + [1 + A \operatorname{cn}(\Omega t, m)] \sin \phi = 0$$

Parametrically damped driven pendulum 62

$$\frac{d^2\phi}{dt^2} + \varepsilon [1 + A \sin(\Omega t)] \frac{d\phi}{dt} + \sin \phi = 0$$

External and parametric periodic forces 73

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + A_1 \cos(\Omega_1 t) \cos \varphi + \sin \phi = A_2 \sin(\Omega_2 t).$$

External random force 81

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi = \xi(t)$$

External constant and random forces 82

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi = a + \xi(t)$$

External periodic and random forces 85

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi = A \cos(\Omega t) + \xi(t)$$

External constant, random and periodic forces 88

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + \sin \phi = a + A \cos(\Omega t) + \xi(t)$$

Parametric random force 89

$$\frac{d^2\phi}{dt^2} + [\omega_0^2 + \xi(t)] \sin \phi = 0$$

Parametric periodic and random force 91

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + [A \sin(\Omega t) + \xi(t)] \sin \phi = 0$$

Parametric periodic and external random force 91

$$\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + [1 + A \sin(\Omega t)] \sin \phi = \xi(t)$$

Parametric random and external constant, periodic and random force 92

$$\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + [\omega_0^2 + \xi(t)] \sin \phi = a + f \sin(\Omega t) + \eta(t)$$

Overdamped pendulum

Additive white noise 93

$$\frac{d\phi}{dt} = a_0 - b_0 \sin \phi + \xi(t)$$

Additive dichotomous noise 96

$$\frac{d\phi}{dt} + b_0 \sin \phi = \xi(t)$$

Multiplicative dichotomous noise 99

$$\frac{d\phi}{dt} = a_0 - [b_0 + \eta(t)] \sin \varphi$$

Additive and multiplicative white noise 102

$$\frac{d\phi}{dt} = [a_0 + \xi(T)] - [b_0 + \eta(t)] \sin \varphi$$

Multiplicative dichotomous and additive white noise 109

$$\frac{d\phi}{dt} = [a_0 + \xi(T)] - [b_0 + \eta(t)] \sin \varphi$$

Correlated additive and multiplicative noise 110

$$\frac{d\phi}{dt} = [a_0 + \xi(T)] - [b_0 + \eta(t)] \sin \varphi$$