Introductory Lectures on **Knot Theory**

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Preface

This book is the fruit of a three-week summer school and conference held at the Abdus Salaam International Centre for Theoretical Physics in Trieste, Italy from May 11 to May 29 of 2009. The editors of this volume were invited to organize the School by Le Dung Trang, then Director of the Mathematics Institute at ICTP. In the three-week summer school, many short courses on knot theory and its applications were given. In this book, we have collected papers that are primarily related to the school. Each paper endeavors to be self-contained and to tell a story about knots from the beginning. In this preface, we shall say a few words about each paper and some words about how the papers relate to one another. We hope that this will provide a guide for the reader of this book.

The program was called the “Advanced School and Conference on Knot Theory and its Applications to Physics and Biology”. There were 21 school lecturers. The conference took place from 25 May to 29 May 2009. There were 62 talks and poster sessions. There were in total 137 participants, most of them young researchers. Among them 41 were female researchers. The participants were from 43 countries. More information, abstracts, school notes and conference material can be found on the ICTP webpage: http://cdsagenda5.ictp.trieste.it/full_display.php?smr=0&ida=a08157. We now give capsule descriptions of the papers in this book.

The paper by Anna Beliakova and Thang Le gives a picture of the invariants of 3-manifolds that have emerged both from Witten’s functional integral, generalized Vassiliev invariants and from the work of Reshetikhin and Turaev via quantum groups. All these constructions of invariants depend upon the use of the Kirby calculus to formulate problems about 3-manifolds in terms of problems about knots and links. This paper contains a quick introduction to the Kirby calculus as well as a survey of invariants of 3-manifolds.

The paper by Scott Carter is an introduction to quandles. Quandles are an algebraic generalization of the fundamental group of the complement of a knot. The algebraic axioms for a quandle parallel the structure of

the Reidemeister moves (a set of moves on knot diagrams that generate isotopies of knots and so reformulate the theory of knots in terms of their diagrams). This paper explores the subject of quandles through to the use of state summations and quandle cocycles and their applications to surfaces embedded in 4-dimensional space.

The paper by Chmutov is an introduction to Vassiliev invariants. This approach to knot invariants was first proposed by Vassiliev in terms of the space of all knots and singular embeddings of knots, but quickly became related to skein theory via Birman and Lin and Lie algebras and physics via Bar-Natan.

The paper by Patrick Dehornoy describes his results on braid-ordering and the motivations of this work in the set theory of large cardinals and relationships with quandles and generalizations of quandles. The reader will see the structural remark that the self-distributivity axiom, placed on a mathematical system with a single binary operation, states that each element of the system acts as a homomorphism of that system to itself. Quandles are self-distributive, but have extra structure. One may ask about systems that are only self-distributive. In early set theoretic investigations, Dehornoy found that models for such single-axiom systems (magmas) could be found by using large transfinite cardinals. Later he found that these ideas could be used and applied in the theory of braids, and the large cardinals were not strictly necessary.

The paper by Roger Fenn gives a different introduction to quandles, starting with the notion of coloring of a knot diagram and continuing to formulations of racks and crossed modules and solutions to the set-theoretic Yang–Baxter equation. Applications are given to classical knots and to welded and virtual knots and links.

The paper by C. McA Gordon is a tight survey of surgery and Dehn fillings for hyperbolic 3-manifolds. This paper can be used as a source of ideas and references for the reader who wants to know more about current methods in the surgery of 3-manifolds.

The paper by Ilyutko and Manturov on graph links is the study of a generalization of knot theory beyond virtual knot theory and related to it via the properties of the intersection graphs of Gauss diagrams for virtual knots. The reader may want to read this paper along with the paper by L. H. Kauffman that introduces virtual knot theory. The graph link generalization of virtual knot theory is very natural, and sheds light on many problems in virtual knot theory. Ilyutko and Manturov take conscientious effort to show examples where their theory is strictly more general than virtual knot theory.

The paper by Slavik Jablan and Radmila Sazdanovic is a survey of open problems related to writhe, unknotting number and the behavior of numerous invariants including properties of Khovanov homology.

The paper by Louis H. Kauffman and Sofia Lambropoulou on unknots and tangles is, in fact, an elementary introduction to the theory of rational tangles, the Conway theorem classifying rational tangles in terms of their fractions, and applications to the structure of unknot diagrams, relationships with the tangle model for DNA recombination and relationships of continued fractions with other areas of mathematics. This paper contains many examples that occurred in the courses taught in Trieste by Kauffman and Lambropoulou.

The paper by Louis H. Kauffman introducing virtual knot theory is an introduction to this diagrammatic theory of knots and links in thickened surfaces and it contains new material about the arrow polynomial of Dye and Kauffman and new material about parity for virtual knots (see also the paper on parity in this volume by V. O. Manturov). The paper contains a quick introduction to Khovanov homology and an even quicker introduction to the Dye–Kauffman–Manturov generalizations of Khovanov homology for the arrow polynomial. Virtual knot theory is a natural extension of classical knot theory and is an active field of research at the time of this writing.

The paper by Louis H. Kauffman introducing Khovanov homology is a first pedagogical step into this large and significant subject. Khovanov homology generalizes the Jones polynomial to a homology theory and is part of a continuing breakthrough in the study of invariants of knots and links.

The paper by Sofia Lambropoulou on braid equivalences and the L -move is a survey of Lambropoulou's use of the L -move (see the paper for this key concept) and its applications to proving Markov theorems for braids in many contexts. The L -move, pioneered by Lambropoulou, provides new and conceptual approaches to the relationships between braids and their closures in many different categories.

The paper by Vassily O. Manturov on free knots and parity is about generalizations of knot theory motivated by virtual knot theory and ways to make new invariants from the inevitable parity considerations that arise in virtual knot theory. In the Gauss diagram of a virtual knot, a given chord may be intersected by an even or by an odd number of other chords. The intersection is always even for classical knots, but in the virtual case and its generalizations, this access to parity leads to a host of new results, many of which are examined in this paper by Manturov.

The paper by Kenneth C. Millett on physical knot theory is a thorough survey of the use of knot theory in the study of polymer chains and of the many problems in the theory of piecewise linear knots that are inspired by the physical questions.

The paper by Hugh R. Morton on knots, satellites and quantum groups is a self-contained introduction to methods for using the HOMFLY polynomial to study satellite knots. In the course of this motivated introduction one meets methods that are related to the symmetric groups and their representations, the construction of link invariants from quantum groups and the use of Kirby calculus to construct 3-manifold invariants. This paper can be read along with the papers by Beliakova and Le and by Pichai in this volume.

The paper by Jozef Przytycki is an introductory survey of the theory of knots and its history. The reader new to knot theory can start with this paper and learn about colorings, tangles, quandles, constructions related to knot diagrams and many other matters fundamental to the theory of knots and links. The reader will also learn how to associate a symplectic structure to a tangle boundary so that tangles become Lagrangians (a proof of this result has not been published before). This paper, starting from history and elementary principles, leads to novel mathematics.

The paper by Ramadevi Pichai is devoted to the detection of chirality and mutations of knots and links via the use of the Chern–Simons–Witten approach to link invariants. This paper gives a quick introduction to the form of Witten’s methods and will serve to whet the reader’s appetite for more about this relationship of knot theory with physical ideas.

The paper by Eric J. Rawdon on physical knot theory should be read in tandem with the paper by Kenneth C. Millett. These two papers give a wide view of this field of application for piecewise linear knots.

The paper by Renzo L. Ricca is an introductory paper on the history and properties of the linking number as defined by C. F. Gauss by an integral related to electromagnetic theory. The paper makes a plausible reconstruction of how Gauss constructed his concept and definition of linking, providing an exciting point of view on the past and present of this central concept for linking and writhing of space curves.

The book also includes the list of participants (p. 503) in the Summer School, the Schedule of Talks and Course of the School (p. 507), a conference photo (p. xii) and a photo of the organizers (p. xiii). A link to the ICTP Advance School and Conference website is as follows: http://cdsagenda5.ictp.trieste.it/display_level.php?fid=1/70

We thank all the participants and lecturers in the Summer School for devoting their energy and time to this course, and we thank the authors of these papers for the extra energy and time that they have devoted to the composition of their work. Still, with all the work in this volume made explicit, this book is just a snapshot of a small part of the flow of information and ideas that went on in Trieste. A dramatic video on irrational Dehn surgery, created during the Trieste meeting, can be found at: http://www.youtube.com/results?search_query=Dehn%27s+dilemma&aq=f

To the reader: We wish you could have been there. If you were there, we hope this volume helps to recapture that time and carry it forward.

The illustration on the back cover of this book was composed by Leonardo DaVinci around 1496. DaVinci is celebrated as a keen observer of Nature, a theorist of powerful and inventive imagination and a very great artist. His depiction of a knot came from his speculations and observations on the nature of space and how space might be filled with a weave. We challenge the reader to compute the Jones polynomial for the DaVinci knot!

On the front cover the reader will find a more recent example of the art of knot depiction by Slavik Jablan. Jablan's work reaches outward into 4-dimensional space from a 3-dimensional weaving pattern. We feel that both illustrations capture the spirit of the book.

ADVANCED SCHOOL AND CONFERENCE ON KNOT THEORY AND ITS APPLICATIONS TO PHYSICS AND BIOLOGY

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