RANDOM SEQUENTIAL PACKING OF CUBES

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To Maja and Yuri

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Preface

A cube packing is a family of translates of the unit cube $[0, 1]^n$ such that the intersection of any two of them has empty interior. If one starts from the empty set and add translates, chosen at random, of the unit cube sequentially until there is no space to pack any more, then one gets a non-extensible cube packing.

The Flory model is perhaps the first, such sequential random packing. It was introduced in Polymer Chemistry to model the behavior of catalyst and it considers integral translates of an interval $[0, 2]$ inside another interval $[0, n]$ and asks for the limit packing density which represents then the efficiency of the catalyst. Another classical such model is the random parking problem, where drivers park their cars of length 1 at random in a parking place interval $[0, x]$. Such models have wide applications to statistical physics.

The first major result in the field is the computation of the limit density C_R of the random parking problem by [Rényi (1958)]. [Palásti (1960)] conjectured that the average density for the sequential random packing of $[0, 1]^d$ in $[0, x]^d$ converges to C_R^d when x goes to ∞ . The existence of the limiting packing density was shown only recently by [Penrose (2001)] but the rigorous computation of the limit packing density is not known for dimension greater than 1. To estimate this d -dimensional limit packing density, a classic method is the sampling procedure or Monte-Carlo simulation of the random sequential packing itself. The results of those simulations do not support Palásti's conjecture on the value of the limit packing density [Blaisdell and Solomon (1970)] and currently the limiting packing density is not known rigorously in dimension $d > 1$.

The 1-dimensional random sequential packing problem can be solved within the framework of classical mathematical analysis. For example in

order to study the recursion formula to get the limiting packing density by [Rényi (1958)], we introduce delay integral equations, for which we apply Laplace transform and then finally a Tauberian theorem which we prove completely. The minimum of gaps gives an interesting nonlinear delay integral equation [Itoh (1980)] whose analysis is amenable by tools of classical analysis which we introduce appropriately when needed. The alternative methodology of [Dvoretzky and Robbins (1964)] is also introduced as well as the related central limit theorem. We consider then a unified Kakutani-Rényi model, which interpolates between the Kakutani interval splitting and the Rényi sequential random packing [Komaki and Itoh (1992)]. We also consider a spin variant of the packing problem, where cars can park with spin-up or spin-down and compute the corresponding limit packing density [Itoh and Shepp (1999)]. By introducing the idea of interval tree, we can make a continuous model of binary search tree which is an important structure in Computer Science [Sibuya and Itoh (1987)]. The tree generated by the sequential bisection has less discrete structure than the original binary search tree and we can make an analogous analysis to the 1-dimensional random packing.

In general d -dimensional random cube packing problem are very hard to analyze mathematically for $d > 1$. Thus simulations are of paramount importance for the understanding of the involved phenomena. We give the general tools for doing random sequential packing by introducing the covering problem, Voronoi diagrams and Delaunay tessellations, which have a lot of applications in Science and Technology. We explain how this works out for the examples of packing by spheres, cubes and cross polytope (unit ball for the Hamming distance in Coding Theory).

We consider the simplest random sequential packing with rigid boundary, i.e. a packing in which cubes of sidelength 2 are put sequentially at random into the cube of sidelength 4, with a cubic grid of unit sidelength [Itoh and Ueda (1983)]. This simplest random sequential packing model seems to have some of the characteristics of the model by Palásti and to be related to the discrete geometry of cubes which has a long history of research going back to Minkowski. Computer Simulations up to dimension 11, suggest that the packing density γ_d satisfies $\gamma_d \simeq d^{-\alpha}$ with an appropriate constant α , as in the case of random sequential coding by Hamming distance [Itoh and Solomon (1986)]. Despite being in many ways a much simpler model, this model is still essentially unsolved. However by using the special geometry of the face lattice of the cube, it is shown that the expected number of decrease of the packing density is less than $(\frac{4}{3})^d$ at

each step of the random sequential packing. This shows that the expected number of cubes at the saturation is larger than $\left(\frac{3}{2}\right)^d$ [Poyarkov (2005)]. We also consider an extension of this model where one packs cubes of size N in a cube of side lengths $2N$.

This packing led us to the introduction of the random sequential packing of cubes of sidelength 2 in a torus of sidelength 4. Just like the previously considered rigid boundary model, this case allows us to introduce a combinatorial formalism to describe the corresponding cube packings. Thus we are able to apply the methods of exhaustive combinatorial enumeration to this setting. This allows us to find a number of remarkable combinatorial structures that could hardly have been found purely by hand computations. Then we consider a continuous variant of this model and we show that again we can describe it by a combinatorial model [Dutour Sikirić and Itoh (2010)]. This allows us to find some new remarkable structures and to find some extremal result on the number of parameters.

Whenever possible, we tried to give a description of the mathematical theory used when they are slightly non-standard and we give appendices on Complex Analysis, Laplace Transform, Renewal Theory, Space Groups and Exhaustive Combinatorial Enumeration. When several methods are possible to solve a problem, for example Tauberian Theorem and Complex Analysis, we have indicated both and discussed the relative merit of the possible approaches.

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