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A Rigorous Approach

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A Rigorous Approach

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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

World Scientific Series on Nonlinear Science, Series A — Vol. 73

2-D QUADRATIC MAPS AND 3-D ODE SYSTEMS

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ISBN-13 978-981-4307-74-1

ISBN-10 981-4307-74-2

Printed in Singapore.

To my family: My father Tayeb and my brothers Zohir, Samra, Mourad, Saida, and Bachlila. To my sweet wife Houda and to all who helped me write this book.

Elhadj

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Preface

This book was written on the basis of research carried out in the past three decades on the rigorous proof of chaos and bifurcations in 2-D quadratic maps, especially the invertible case of the Hénon map, and on 3-D piecewise linear ODEs, especially Chua's circuit. In addition, we have recently published work in the field of general 2-D quadratic maps, especially their classification into equivalence classes and finding regions of chaos, hyperchaos, and nonchaos in the space of bifurcation parameters.

The first chapter is the main introduction to the rigorous tools used to prove chaos and bifurcations in the two representative systems. Chapter 2 is focused on the study of the invertible case of the 2-D quadratic map, where the most significant works in the literature are oriented toward the Hénon mapping. Chapter 3 classifies the 2-D quadratic case into 30 maps with well known formulas. Two proofs for the regions of chaos, hyperchaos, and nonchaos in the space of bifurcation parameters are provided using a technique based on the second-derivative test and bounds for Lyapunov exponents. Chapter 4 is concerned with the rigorous proof of chaos in the piecewise linear Chua's system using two methods, the first of which is based on construction of the Poincaré map, and the second is based on a computer-assisted proof. Finally, Chapter 5 discusses the rigorous analysis of bifurcation phenomena in the piecewise linear Chua's system using both an analytical 2-D and a 1-D approximate Poincaré mapping in addition to other analytical methods.

Zeraoulia Elhadj
Julien Clinton Sprott
March 2010

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Acknowledgements

We thank L.O.Chua, John Milnor, Frederick Marotto, Michal Misiurewicz, Leonardo Mora, Carsten Knudsen, Huseyin Kocak, Peter Grassberger, Ernest Fontich, Zbigniew Nitecki, Zbigniew Galias, and Piotr Zgliczynski for providing some of their old papers concerning the topics in this book.