**Major American Universities Ph.D.
Qualifying Questions and Solutions – Mathematics**

Problems and Solutions in Mathematics

Second Edition

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Preface

This book covers six aspects of graduate school mathematics: Algebra, Topology, Differential Geometry, Real Analysis, Complex Analysis and Partial Differential Equations. It contains a selection of more than 500 problems and solutions based on the Ph.D. qualifying test papers of a decade of influential universities in North America. The mathematical problems under discussion are kept within the scope of the textbooks for graduate students. Finding solutions to these problems, however, involves a deep understanding of mathematical principles as well as an acquisition of skills in analysis and computation. As a supplement to textbooks, this book may prove to be of some help to the students in taking relevant courses. It may also serve as a reference book for the teachers concerned.

It has to be pointed out that this book should not be regarded as an all-purpose troubleshooter. Nor is it advisable to take the book as an exemplary text and commit to memory all the problems and solutions and make an indiscriminate use of them. Instead, the students are expected to make a selective survey of the problems, take a do-it-yourself approach and arrive at their own solutions which they may check against those listed in the book. It would be gratifying to see that the students can work out the problems on their own and come up with better solutions than those provided by the book. If the students fail to do so or their solutions may turn out to be incomplete, it may reveal the inadequacy of their knowledge or approach, thus spurring them to greater efforts to promote their skills. The very purpose of

the authors in writing the book is just to help the students to discover the truth by trial and error.

This book was inspired by Professor K. K. Phua's proposals. We are particularly grateful to him for his support. We also wish to thank Dr. Xu Pei-jun, Professors Zhang Yin-nan, Hong Jia-xing and Chen Xiao-man for their painstaking efforts to collect test-oriented data. For selecting problems and providing solutions, we wish to acknowledge the following professors respectively: Wu Quan-shui (Part I), Pan Yang-lian (Part II), Jiang Guo-ying (Part III), Tong Yu-sun, Xu Sheng-zhi (Part IV), Chen Ji-xiu (Part V) and Qin Tie-hu (Part VI). We are also indebted to Professor Guo Yu-tao for carefully reading and correcting the manuscript. Finally, we pay tribute to Dr. Cai Zhi-jie for printing out the manuscript.

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Preface to the Second Edition

It has been twelve years since the first publication of this book. During the period, new mathematical problems, including many classical and interesting ones, kept emerging in the Ph.D. Qualifying Test of some prestigious universities in North America. For the improvement of this book and with the encouragement of World Scientific Publishing Co., we carefully selected more than one hundred new mathematical problems from the Ph.D. Qualifying Test papers in the past decade and provided solutions to them respectively, which can be taken as a necessary supplement to the first edition. Besides, we also made suitable corrections or amendments to the first edition regarding the solutions to a few problems which were not written in a very clear way or contained loopholes. We hope that the second edition of the book will be more helpful to readers.

Finally, on this occasion, we would like to express our gratitude to the readers for their concern and to the World Scientific for their support.

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