Truly *Nonlinear* Oscillations

Harmonic Balance, Parameter Expansions,
Iteration, and Averaging Methods

This page intentionally left blank

Truly, *Nonlinear* Oscillations

Harmonic Balance, Parameter Expansions,
Iteration, and Averaging Methods

Ronald E Mickens
Clark Atlanta University, USA

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Mickens, Ronald E., 1943–

Truly nonlinear oscillations : harmonic balance, parameter expansions, iteration,
and averaging methods / by Ronald E. Mickens.

p. cm.

Includes bibliographical references and index.

ISBN-13: 978-981-4291-65-1 (hardcover : alk. paper)

ISBN-10: 981-4291-65-X (hardcover : alk. paper)

1. Approximation theory. 2. Nonlinear oscillations. I. Title.

QA221.M53 2010

511'.4--dc22

2009038794

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Copyright © 2010 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

Printed in Singapore.

Truly Nonlinear Oscillators

An Introduction to Harmonic Balance, Parameter Expansion,
Iteration, and Averaging Methods

Ronald E. Mickens

Distinguished Fuller E. Callaway Professor

Clark Atlanta University

Atlanta, GA 30314, USA

This book is dedicated to my family:
Daughter ... Leah,
Son ... James,
Wife ... Maria.

Preface

This small volume introduces several important methods for calculating approximations to the periodic solutions of “truly nonlinear” (TNL) oscillator differential equations. This class of equations take the form

$$\ddot{x} + g(x) = \epsilon F(x, \dot{x}),$$

where $g(x)$ has no linear approximation at $x = 0$. During the past several decades a broad range of calculational procedures for solving such differential equations have been created by an internationally based group of researchers. These techniques appear under headings such as

- averaging
- combined and linearization
- harmonic balance
- homotopy perturbation
- iteration
- parameter expansion
- variational iteration methods.

Further, these methodologies have not only been applied to TNL oscillators, but also to strongly nonlinear oscillations where a parameter may take on large values. Most of these techniques have undergone Darwinian type evolution and, as a consequence, a large number of papers are published each year on specializations of a particular method. While we have been thorough in our personal examination of the research literature, only those papers having an immediate connection to the topic under discussion are cited because of the magnitude of existing publications and because an interested user of this volume can easily locate the relevant materials from various websites.

We have written this book for the individual who wishes to learn, understand, and apply available techniques for analyzing and solving problems involving TNL oscillations. It is assumed that the reader of this volume has a background preparation that includes knowledge of perturbation methods for the standard oscillatory systems modeled by the equation

$$\ddot{x} + x = \epsilon F(x, \dot{x}).$$

In particular, this includes an understanding of concepts such as secular terms, limit-cycles, uniformly-valid approximations, and the elements of Fourier series.

The basic style and presentation of the material in this book is heuristic rather than rigorous. The references at the end of each chapter, along with an examination of relevant websites, will allow the reader to fully comprehend what is currently known about a particular technique. However, the reader should also realize that the creation and development of most of the methods discussed in this book do not derive from rigorous mathematical derivations. This task is a future project for those who have the interests and necessary background to carry out these procedures. However, these efforts are clearly not relevant for our present needs.

The book consists of seven chapters and several appendices. Chapter 1 offers an overview of the book. In particular, it presents a definition of TNL equations, introduces the concept of odd-parity systems, and calculates the exact solutions to four TNL oscillatory systems.

Chapter 2 provides a brief discussion of several procedures for a priori determining whether a given TNL differential equation has periodic and/or oscillatory solutions. The next four chapters present introductions to most of the significant procedures for calculating analytical approximations to the solutions of TNL differential equations. These chapters discuss, respectively, harmonic balance, parameter expansion, iteration, and averaging methods. Each chapter gives not only the basic methodology for each technique, but also provides a range of worked examples illustrating their use.

The last chapter considers six TNL oscillator equations and compares results obtained by all the methods that are applicable to each. It ends with general comments on TNL oscillators and provides a short listing of unresolved research problems.

We also include a number of appendices covering topics relevant to understanding the general issues covered in this book. The topics discussed range from certain mathematical relations to basic results on linear second-order differential equations having constant coefficients. Brief presentations

are given on Fourier series, the Lindstedt-Poincaré perturbation method, and the standard first-order method of averaging. A final appendix, “Discrete Models of Two TNL Oscillators,” illustrates the complexities that may arise when one attempts to construct discretizations to calculate numerical solutions.

I thank my many colleagues around the world for the interest in my work, their generalization of these results and their own original “creations” on the subject of TNL oscillations. As always, I am truly grateful to Ms. Annette Rohrs for her technical services in seeing that my handwritten pages were transformed into the present format. Both she and my wife, Maria Mickens, provided valuable editorial assistance and the needed encouragement to successfully complete this project. Finally, I wish to acknowledge Dr. Shirley Williams-Kirksey, Dean of the School of Arts and Sciences, for providing Professional Development Funds to assist in the completion of this project. Without this support the writing effort would not have been done on time.

Ronald E. Mickens
Atlanta, GA
August 2009

This page intentionally left blank

Contents

<i>Preface</i>	vii
<i>List of Figures</i>	xix
<i>List of Tables</i>	xxi
1. Background and General Comments	1
1.1 Truly Nonlinear Functions	1
1.2 Truly Nonlinear Oscillators	2
1.3 General Remarks	3
1.4 Scaling and Dimensionless Form of Differential Equations	5
1.4.1 Linear Damped Oscillator	5
1.4.2 Nonlinear Oscillator	6
1.4.3 $\ddot{x} + ax^p = 0$	7
1.4.4 $\ddot{x} + ax + bx^{1/3} = 0$	8
1.5 Exactly Solvable TNL Oscillators	9
1.5.1 Antisymmetric, Constant Force Oscillator	10
1.5.2 Particle-in-a-Box	11
1.5.3 Restricted Duffing Equation	12
1.5.4 Quadratic Oscillator	14
1.6 Overview of TNL Oscillator Methods	14
1.6.1 Harmonic Balance	16
1.6.2 Parameter Expansion	16
1.6.3 Averaging Methods	17
1.6.4 Iteration Techniques	18
1.7 Discussion	18
Problems	20

References	21
2. Establishing Periodicity	23
2.1 Phase-Space	23
2.1.1 System Equations	24
2.1.2 Fixed-Points	24
2.1.3 ODE for Phase-Space Trajectories	25
2.1.4 Null-clines	25
2.1.5 Symmetry Transformations	26
2.1.6 Closed Phase-Space Trajectories	26
2.1.7 First-Integrals	26
2.2 Application of Phase-Space Methods	27
2.2.1 Linear Harmonic Oscillator	27
2.2.2 Several TNL Oscillator Equations	31
2.3 Dissipative Systems: Energy Methods	33
2.3.1 Damped Linear Oscillator	35
2.3.2 Damped TNL Oscillator	35
2.3.3 Mixed-Damped TNL Oscillator	36
2.4 Resume	39
Problems	39
References	40
3. Harmonic Balance	43
3.1 Direct Harmonic Balance: Methodology	44
3.2 Worked Examples	46
3.2.1 $\ddot{x} + x^3 = 0$	47
3.2.2 $\ddot{x} + x^{-1} = 0$	49
3.2.3 $\ddot{x} + x^2 \text{sgn}(x) = 0$	51
3.2.4 $\ddot{x} + x^{1/3} = 0$	54
3.2.5 $\ddot{x} + x^{-1/3} = 0$	57
3.3 Rational Approximations	61
3.3.1 Fourier Expansion	62
3.3.2 Properties of a_k	62
3.3.3 Calculation of $\ddot{x}$	63
3.4 Worked Examples	63
3.4.1 $\ddot{x} + x^3 = 0$	63
3.4.2 $\ddot{x} + x^2 \text{sgn}(x) = 0$	65
3.4.3 $\ddot{x} + x^{-1} = 0$	66

3.5	Third-Order Equations	67
3.5.1	Castor Model	68
3.5.2	TNL Castor Models	69
3.6	Resume	70
3.6.1	Advantages	70
3.6.2	Disadvantages	70
	Problems	71
	References	72
4.	Parameter Expansions	75
4.1	Introduction	75
4.2	Worked Examples	76
4.2.1	$\ddot{x} + x^3 = 0$	76
4.2.2	$\ddot{x} + x^{-1} = 0$	78
4.2.3	$\ddot{x} + x^3/(1 + x^2) = 0$	80
4.2.4	$\ddot{x} + x^{1/3} = 0$	81
4.2.5	$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}$	84
4.2.6	$\ddot{x} + \text{sgn}(x) = 0$	85
4.3	Discussion	86
4.3.1	Advantages	87
4.3.2	Difficulties	87
	Problems	87
	References	88
5.	Iteration Methods	89
5.1	General Methodology	89
5.1.1	Direct Iteration	89
5.1.2	Extended Iteration	91
5.2	Worked Examples: Direct Iteration	92
5.2.1	$\dot{x} + x^3 = 0$	92
5.2.2	$\ddot{x} + x^3/(1 + x^2) = 0$	97
5.2.3	$\ddot{x} + x^{-1} = 0$	100
5.2.4	$\ddot{x} + \text{sgn}(x) = 0$	103
5.2.5	$\ddot{x} + x^{1/3} = 0$	105
5.2.6	$\ddot{x} + x^{-1/3} = 0$	108
5.2.7	$\ddot{x} + x + x^{1/3} = 0$	110
5.3	Worked Examples: Extended Iteration	112
5.3.1	$\ddot{x} + x^3 = 0$	113

5.3.2	$\ddot{x} + x^{-1} = 0$	115
5.4	Discussion	117
5.4.1	Advantages of Iteration Methods	118
5.4.2	Disadvantages of Iteration Methods	119
	Problems	120
	References	121
6.	Averaging Methods	123
6.1	Elementary TNL Averaging Methods	124
6.1.1	Mickens-Oyedeki Procedure	124
6.1.2	Combined Linearization and Averaging Method	126
6.2	Worked Examples	129
6.2.1	$\ddot{x} + x^3 = -2\epsilon\dot{x}$	129
6.2.2	$\ddot{x} + x^3 = -\epsilon\dot{x}^3$	131
6.2.3	$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}$	132
6.2.4	$\ddot{x} + x^{1/3} = -2\epsilon\dot{x}$	133
6.2.5	$\ddot{x} + x^{1/3} = \epsilon(1 - x^2)\dot{x}$	134
6.2.6	$\ddot{x} + x = -2\epsilon(\dot{x})^{1/3}$	135
6.2.7	General Comments	137
6.3	Cveticanin's Averaging Method	138
6.3.1	Exact Period	139
6.3.2	Averaging Method	140
6.3.3	Summary	142
6.4	Worked Examples	142
6.4.1	$\ddot{x} + x x ^{\alpha-1} = -2\epsilon\dot{x}$	142
6.4.2	$\ddot{x} + x x ^{\alpha-1} = -2\epsilon(\dot{x})^3$	144
6.4.3	$\ddot{x} + x x ^{\alpha-1} = \epsilon(1 - x^2)\dot{x}$	145
6.5	Chronology of Averaging Methods	147
6.6	Comments	149
	Problems	151
	References	152
7.	Comparative Analysis	155
7.1	Purpose	155
7.2	$\ddot{x} + x^3 = 0$	156
7.2.1	Harmonic Balance	156
7.2.2	Parameter Expansion	158
7.2.3	Iteration	158

7.2.4	Comments	159
7.3	$\ddot{x} + x^{1/3} = 0$	160
7.3.1	Harmonic Balance	160
7.3.2	Parameter Expansion	161
7.3.3	Iteration	162
7.3.4	Comment	162
7.4	$\ddot{x} + x^3 = -2\epsilon\dot{x}$	163
7.4.1	Mickens-Oyedjeji	163
7.4.2	Combined-Linearization-Averaging	165
7.4.3	Cveticanin's Method	166
7.4.4	Discussion	167
7.5	$\ddot{x} + x^{1/3} = -2\epsilon\dot{x}$	167
7.5.1	Combined-Linearization-Averaging	167
7.5.2	Cveticanin's Method	168
7.5.3	Discussion	170
7.6	$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}$	170
7.6.1	Mickens-Oyedjeji	170
7.6.2	Cveticanin's Method	171
7.6.3	Discussion	172
7.7	$\ddot{x} + x^{1/3} = \epsilon(1 - x^2)\dot{x}$	175
7.8	General Comments and Calculation Strategies	175
7.8.1	General Comments	176
7.8.2	Calculation Strategies	177
7.9	Research Problems	179
	References	181
	Appendix A Mathematical Relations	183
A.1	Trigonometric Relations	183
A.1.1	Exponential Definitions of Trigonometric Functions	183
A.1.2	Functions of Sums of Angles	183
A.1.3	Powers of Trigonometric Functions	183
A.1.4	Other Trigonometric Relations	184
A.1.5	Derivatives and Integrals of Trigonometric Functions	185
A.2	Factors and Expansions	186
A.3	Quadratic Equations	187
A.4	Cubic Equations	187
A.5	Differentiation of a Definite Integral with Respect to a Parameter	188

A.6	Eigenvalues of a 2×2 Matrix	188
	References	189
Appendix B Gamma and Beta Functions		191
B.1	Gamma Function	191
B.2	The Beta Function	191
B.3	Two Useful Integrals	192
Appendix C Fourier Series		193
C.1	Definition of Fourier Series	193
C.2	Convergence of Fourier Series	194
	C.2.1 Examples	194
	C.2.2 Convergence Theorem	194
C.3	Bounds on Fourier Coefficients	195
C.4	Expansion of $F(a \cos x, -a \sin x)$ in a Fourier Series	195
C.5	Fourier Series for $(\cos \theta)^\alpha$ and $(\sin \theta)^\alpha$	196
	References	198
Appendix D Basic Theorems of the Theory of Second-Order Differential Equations		199
D.1	Introduction	199
D.2	Existence and Uniqueness of the Solution	200
D.3	Dependence of the Solution on Initial Conditions	200
D.4	Dependence of the Solution on a Parameter	201
	References	202
Appendix E Linear Second-Order Differential Equations		203
E.1	Basic Existence Theorem	203
E.2	Homogeneous Linear Differential Equations	203
	E.2.1 Linear Combination	204
	E.2.2 Linear Dependent and Linear Independent Functions	204
	E.2.3 Theorems on Linear Second-Order Homogeneous Differential Equations	204
E.3	Inhomogeneous Linear Differential Equations	205
	E.3.1 Principle of Superposition	206
	E.3.2 Solutions of Linear Inhomogeneous Differential Equations	207

E.4	Linear Second-Order Homogeneous Differential Equations with Constant Coefficients	207
E.5	Linear Second-Order Inhomogeneous Differential Equations with Constant Coefficients	208
E.6	Secular Terms	210
	References	211
Appendix F	Lindstedt-Poincaré Perturbation Method	213
	References	216
Appendix G	A Standard Averaging Method	217
	References	220
Appendix H	Discrete Models of Two TNL Oscillators	221
H.1	NSFD Rules	221
H.2	Discrete Energy Function	222
H.3	Cube-Root Equation	223
H.4	Cube-Root/van der Pol Equation	225
	References	226
	<i>Bibliography</i>	227
	<i>Index</i>	237

This page intentionally left blank

List of Figures

2.2.1 Basic properties of the phase-plane for the linear harmonic oscillator. The dashed-line (- -) is the “zero” null-cline, the solid line (—) is the “infinite” null-cline. The ($\pm$) indicates the sign of dy/dx for the designated region.	29
2.2.2 Geometric proof that phase-plane trajectories are closed curves using the symmetry transformations.	30
2.3.1 $\ddot{x} + f(x) = -\epsilon g(x)\dot{x}$: (a) Periodic solutions for $\epsilon = 0$. (b) Damped oscillatory solutions for $\epsilon > 0$	36
2.3.2 Phase-plane for Eq. (2.3.15). The dashed lines are the $y_0(x)$ null-clines. The solid line is the $y_\infty(x)$ null-cline.	38
2.3.3 Typical trajectories in the phase-plane for Eq. (2.3.17).	38
3.2.1 (a) Phase-plane for $\ddot{x} + x^{-1/3} = 0$. Vertical dashes denote the infinite null-cline, $y_\infty(x)$. (b) Trajectory passing through $x(0) = A$ and $y(0) = 0$	58
5.2.1 Plot of $\Omega^2(A)$ versus A for the periodic solutions of Eq. (5.2.118).	113
6.2.1 Schematic representation of the solution for $\ddot{x} + x = -2\epsilon(\dot{x})^{1/3}$	137
7.4.1 Plot of $\Omega(t, A, \epsilon)$ versus t , for the linearly damped, pure cubic Duffing equation. $\Omega_0(A) = (\frac{3}{4})^{1/2} A$	164
7.4.2 Plot of the numerical solution of $\ddot{x} + x^3 = -2\epsilon\dot{x}$ for $\epsilon = 0.01$, $x(0) = 1$ and $\dot{x}(0) = 0$	165
7.5.1 Plot of the numerical solution of $\ddot{x} + x^{1/3} = -2\epsilon\dot{x}$ for $\epsilon = 0.01$, $x(0) = 1$ and $\dot{x}(0) = 0$	168

7.5.2 This graph is the same as that in Figure 7.5.1, except that the interval in time is twice as long.	169
7.6.1 Numerical solution of Eq. (7.6.1) for $x(0) = 4$, $y(0) = 0$, and $\epsilon = 0.1$	173
7.6.2 Numerical solution of Eq. (7.6.2) for $x(0) = 0.1$, $y(0) = 0$, and $\epsilon = 0.1$	174

List of Tables

3.2.1 Values for $A^{1/3}\Omega(A)$	57
7.4.1 Comparison of the amplitude and effective angular frequencies for the linearly damped, pure cubic, Duffing oscillator.	167