NUMERICAL CONFORMAL MAPPING

Domain Decomposition and the
Mapping of Quadrilaterals

This page intentionally left blank

NUMERICAL CONFORMAL MAPPING

Domain Decomposition and the
Mapping of Quadrilaterals

Nicolas Papamichael

University of Cyprus, Cyprus

Nikos Stylianopoulos

University of Cyprus, Cyprus

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

NUMERICAL CONFORMAL MAPPING

Domain Decomposition and the Mapping of Quadrilaterals

Copyright © 2010 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

ISBN-13 978-981-4289-52-8

ISBN-10 981-4289-52-3

Printed in Singapore.

Dedicated to the memory of Professor Dieter Gaier

This page intentionally left blank

Preface

Let Q be a quadrilateral consisting of a Jordan domain Ω and four specified points z_1, z_2, z_3, z_4 in counterclockwise order on its boundary, and let $\mathcal{R}$ be a rectangle of base length a and height b . Also, let h be the aspect ratio of $\mathcal{R}$, i.e. $h := b/a$. Then, the conformal module $m(Q)$ of Q is the unique value of h for which Q is conformally equivalent to the rectangular quadrilateral consisting of $\mathcal{R}$ and its four vertices. By this it is meant that for $h = m(Q)$, and for this value only, there exists a conformal mapping $F : \Omega \rightarrow \mathcal{R}$, of Ω onto $\mathcal{R}$, such that its extension to $\partial\Omega$ takes the four specified points z_1, z_2, z_3, z_4 of Q , respectively, onto the four vertices of $\mathcal{R}$.

Apart from being an important domain functional from the function theoretic point of view, the conformal module of a quadrilateral is also intimately related to certain physical constants that occur in engineering applications. In particular, $m(Q)$ plays a very central role in applications involving the measurement of resistance values of integrated circuit networks. For these reasons, the problem of determining $m(Q)$ and F is of considerable interest both from the theoretical and the practical points of view.

This book is intended as a monograph on numerical conformal mapping, but with the emphasis on the computational and application aspects of the problem of determining $m(Q)$ and the associated conformal mapping $F : \Omega \rightarrow \mathcal{R}$. In writing it we have sought to present a comprehensive study of the properties and areas of application of $m(Q)$, to outline the available numerical methods for computing both $m(Q)$ and F and, at the same time, to provide information about the existing numerical methods and associated software for approximating some of the more standard conformal mappings. However, the book has been written primarily for the following two reasons:

- (1) To provide a detailed and self-contained account of the theory of a domain decomposition method (DDM), which has been developed in recent years ([42], [43], [62], [63], [117], [147]–[152]) for the purpose of computing the conformal modules and associated conformal mappings of elongated quadrilaterals.
- (2) To illustrate how the DDM can be used to compute accurate approximations to the conformal modules and associated conformal mappings of complicated quadrilaterals, of the type that occur frequently in engineering applications.

The material is organized as follows:

Chapter 1 is a preparatory chapter containing basic results about the standard conformal mapping problems for simply and doubly-connected domains, and details of the available numerical methods and associated software for the computation of these mappings.

In Chapter 2 we study the properties and areas of application of conformal modules, and also consider some of the available numerical methods for computing approximations to $m(Q)$ and F . In particular, this chapter includes the following:

- A detailed description of the “conventional method”, i.e. of the standard approach of seeking to determine $m(Q)$ and $F : \Omega \rightarrow \mathcal{R}$ after first determining the conformal mapping of Ω onto the unit disc, or the upper half-plane.
- A detailed study of a serious numerical drawback that affects adversely the numerical implementation of the conventional method. This drawback is due to the so-called “crowding phenomenon”, i.e. to the crowding of points on the unit circle, or the real axis, which is caused (when the quadrilateral under consideration is “long”) by the intermediate conformal mapping that takes Ω onto the unit disc, or the upper-half plane.
- Brief discussions of various numerical methods and, in particular, of two methods (due to Howell and Trefethen [100] and Driscoll and Vavasis [38]) that have been devised specifically for the purpose of overcoming the crowding difficulties associated with the conventional method.

Chapters 3 and 4 are devoted to the theory and application of the DDM. Chapter 3 is concerned with the mapping of a special class of quadrilaterals, where: (i) the defining domain is bounded by two parallel straight lines and two Jordan arcs, and (ii) the four specified points are the corners where the

two arcs intersect the straight lines. The chapter includes a discussion of the original DDM theory, which was introduced by us in [147] and [148], and a study of the corresponding refined theories of Gaier and Hayman [62], [63] and Laugesen [117].

Finally, Chapter 4 is based on results contained in a series of papers ([149]–[152] by us and [42], [43] by M.I. Falcão and us), and concerns the general DDM theory and other related matters, including the use of the method for the measurement of resistance values of integrated circuit networks. The chapter contains, in particular, a study of certain important results on the asymptotic behavior of conformal modules, a discussion of the quality of various engineering rules for computing conformal modules, and the description of a very simple DDM procedure for computing (often by hand calculation) the conformal modules of meander-like polygonal quadrilaterals of the type that occur frequently in applications. The chapter also contains several numerical examples illustrating the use of the DDM (in conjunction with some of the available computer software for mapping onto the unit disc) for computing accurate approximations to the conformal modules and associated conformal mappings of complicated quadrilaterals.

The book is addressed primarily to applied mathematicians and numerical analysts, who are interested in numerical conformal mapping and wish to find out more about some of the recent developments of the subject, and to practicing engineers and applied scientists who use (or would like to use) conformal transformations and wish to find out more about the capabilities of modern numerical conformal mapping. At the same time, the book should be of interest to more theoretically oriented mathematicians, because it contains material that: (i) highlight the crucial role that function theory plays in the development of the numerical methods, and (ii) illustrate the theoretical insight that can, frequently, be gained from the results of numerical experiments. More generally, the book could serve as a supplementary graduate text on the properties and areas of application of conformal modules, and as a guide with up to date information on the capabilities of some of the available numerical methods and associated software for conformal mapping.

Regarding prerequisites, we sought to keep these to a minimum, so as to make the book accessible to all those who have a basic knowledge of function theory and, in particular, to engineers and scientists whose mathematical training included a course in complex variables. For this reason, although we have tried to give a complete and rigorous treatment of the theory of the domain decomposition method, there were instances

(especially in Chapter 3) where we have been forced to make use of certain deeper mathematical results without presenting their detailed proofs. In all such instances, however, we have given specific guidelines of where the proofs can be found. In addition, in cases where the results had to undergo modifications before being used in our work, we have given the details needed for deriving these modifications.

All definitions, theorems, lemmas, corollaries and equations are numbered consecutively, with the first two digits of the number indicating, respectively, the chapter and the section in which the particular item occurs. Also, for ease of back-referencing, the discussion of the results of a particular section is frequently broken down into a list of remarks which are numbered in the same fashion.

The book is dedicated to the memory of Professor Dieter Gaier, as a modest tribute to his contribution to the development of modern numerical conformal mapping, and as an expression of gratitude for his generous support of our work over the years.

Nicosia, Cyprus,
October 2009

N. Papamichael and N.S. Stylianopoulos

Contents

<i>Preface</i>	vii
1. Standard Conformal Mappings and Software	1
1.1 Introduction	1
1.2 The mapping of simply-connected domains	2
1.3 The mapping of doubly-connected domains	9
1.4 Numerical conformal mapping	13
1.5 The Bergman kernel method	16
1.6 The integral equation method of Symm	21
1.7 Schwarz-Christoffel mappings	25
1.8 Conjugate periodic functions on an annulus	31
1.9 The method of Garrick	35
1.10 Numerical examples	44
2. Conformal Modules of Quadrilaterals	51
2.1 Basic definitions and properties	51
2.2 Physical interpretation and applications	55
2.3 Further properties	61
2.4 The conventional method	71
2.5 The crowding phenomenon	75
2.6 Examples	80
2.7 A finite-element method	86
2.8 A modified Schwarz-Christoffel method	88
2.9 Cross-ratios and Delaunay triangulation (CRDT)	90
2.10 The method of Garrick for special quadrilaterals	91

2.11	The use of Laplacian solvers	100
3.	Domain Decomposition for Special Quadrilaterals	105
3.1	Introduction	105
3.2	Error estimation via the theory of the Garrick method . .	107
3.3	Error estimates for conformal modules	116
3.4	Error estimates for the full conformal mapping	130
4.	Domain Decomposition for General Quadrilaterals	141
4.1	Introduction	141
4.2	Error estimates for conformal modules	142
4.3	Curvilinear crosscuts	154
4.4	The asymptotic behavior of conformal modules	166
4.5	Engineering rules for conformal modules	177
4.6	Error estimates for the full conformal mapping	187
	<i>Bibliography</i>	211
	<i>Index</i>	227