NONSTANDARD METHODS IN FUNCTIONAL ANALYSIS

lectures and notes

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Printed in Singapore.

To my dearest MOTHER

—my first Teacher

from whom I learnt

一山還比一山高

⁰Translation: *Any high peak is yet to be humbled by a higher peak somewhere.*

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Preface

In the early 1960s, using techniques from the model theory of first order logic, Robinson gave a rigorous formulation and extension of Leibniz infinitesimal calculus. Since then the methodology has found application in a wide spectrum of areas in mathematics, with particular success in probability theory and functional analysis. In the latter, fruitful results were produced with Luxemburg's invention of the nonstandard hull construction.

There is so far no publication of a coherent and self-contained treatment of functional analysis using methods from nonstandard analysis. Therefore, this publication seeks to fill such a gap.

In a way, writing a book like this is tantamount to writing a fantasy novel on a plausible alternative evolution of mathematics: *What if rigorous nonstandard analysis were invented and become popularized before the development of Banach space theory?*

However, by adhering to such a theme too dogmatically, one misses out lots of excitement—as it is unwise to prescribe a methodology before investigating the problems. For that reason, the purpose of this book is simply to demonstrate how intuition and methods from both the classical camp and the nonstandard camp are brought together to create the fundamental concepts and results in functional analysis.

READERSHIP

This book aims at both senior/graduate level students and researchers in functional analysis. For the former, it can be used as a self-study aid or a textbook for a course in functional analysis. For the latter, the book can be used as a reference for techniques from nonstandard analysis applicable to functional analysis, as well as for directions to further research.

PREREQUISITE

Undergraduate level courses covering naive set theory, real analysis, complex analysis and preferably some basic measure theory.

SYNOPSIS

Chapter 1 (p.1–75): A brief introduction to the logical and set-theoretical framework needed for nonstandard analysis is first presented in §1.1. Then two constructions of the nonstandard universe are given in §1.2 and §1.3. In §1.4, as a warm-up exercise, elementary calculus is used as a testing ground for convincing the reader that nonstandard analysis is indeed a collection of simple, effective and intuitive mathematical tools. §1.5 and §1.6 continue to serve the same purpose and present in a nonstandard manner all measure-theoretical and topological background required in later chapters.

§1.1–1.4 can be skipped by those with requisite skills in nonstandard analysis, namely being fluent with the transfer principle, the saturation principle, internal sets and internal extensions. §1.5 and §1.6 can be skipped by those who already had a thorough introduction to measure theory and topology, although they may still enjoy browsing through the nonstandard treatment of such topics.

Chapter 2 (p.77–180): In §2.1 and §2.2 basic results concerning normed linear spaces, Banach spaces, linear operators and the nonstandard hull construction are given. In §2.3, Helly's Theorem is placed in the nonstandard context and regarded as the most fundamental result in Banach space theory. The nonstandard hull construction is no doubt the most central notion in this book. A version of this construction is applied in §2.4. to represent the bidual of a Banach space. Reflexive spaces are perhaps the most studied class of Banach spaces. They are dealt with in §2.5. In §2.6 Hilbert spaces are given just sufficient coverage for the operator theory in the next chapter. §2.7 consists of a selection of topics, including the invariant subspace problem which witnessed the success of nonstandard analysis in the early days of its development.

Chapter 3 (p.181–275): In §3.1 Banach algebras and spectra are introduced. The nonstandard hull construction is extended in this context. C^* -algebras, Gelfand transform and the GNS construction are handled in §3.2. The norm-nonstandard hull of a C^* -algebra is the topic of §3.3. As a prominent type of C^* -algebras, von Neumann algebras are featured in §3.4 with a study of the effect of various kinds of nonstandard hull constructions on them. §3.5 is about some aspects and usage of projections.

Chapter 4 (p.277–301): This chapter is made up of some new results in Banach space and Banach algebras, such as isometric identities for Hilbert space-valued integrals, fixed point theorems, representation of Arens product on a bidual and a noncommutative version of Loeb measures. Some tangible open problems and questions are listed with the intention of inviting the serious reader to make contribution to the advancement of nonstandard methods.

COURSE TOPICS

One-semester senior level course: Chapter 1 (§1.3 may substitute part of §1.2.2), §2.1–2.6.

Two-semester senior level course or one-semester graduate level course: The above, together with §2.7, §3.1–3.4 and any part of §3.5 and Chapter 4. Possibly with more detail on logic and ultraproduct supplemented by Chang and Keisler (1990).

Whenever possible, exercise problems should be done by using methods from nonstandard analysis.

ATTRIBUTIONS

Other than the author's negligence, the absence of ascribing credit for a result means that it is easy or folklore or due to the author.

ACKNOWLEDGEMENT

I am indebted to Hua and Yisun for their patience and tolerance during the writing of this book. I also thank Hua for the cover design and drawings.

Siu-Ah Ng
January 2010

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Convention and Symbols

- A *Banach space, linear functional, etc.* is an object which is a Banach space, linear functional, *etc.* in the ordinary (*standard*) sense.
- An *internal Banach space, internal linear functional, etc.* is an *internal* object which is a $\ast$ Banach space, $\ast$ linear functional, *etc.*
- Notation evolves along the pages as topics become more specialized. For example, the generic notation for a linear operator on a Hilbert space changes from f to ϕ then to T and sometimes may reverse, depending on the context.
- In the first half of the book, the symbol $\ast$ is more rigorously attached to any internal extension. In the second half, this practice is eased out for better readability. So $\ast\widehat{\mathcal{B}(\ast H)}$ can be written as $\widehat{\mathcal{B}(H)}$.
- From p.58 onward, unless stated otherwise, all topological spaces are assumed to be *Hausdorff*.
- By $\forall i \in I (\dots)$ holds, we mean $(\dots)$ holds for every $i \in I$. We also use $\exists i \in I (\dots)$ in a similar way. The conjunction $\bigwedge_{i \in I} (\dots)$ is equivalent to $\forall i \in I (\dots)$ and the disjunction $\bigvee_{i \in I} (\dots)$ is equivalent to $\exists i \in I (\dots)$. None of these are first order formulas if I is infinite.
- $X := (\dots)$ means X is defined as $(\dots)$.
- Respectively, $\subset$ and $\supset$ mean $\subseteq$ and $\supseteq$, but $<$ and $>$ mean $\lessdot$ and $\gtrdot$.
- The image of a set X under a function f is denoted by $f[X]$.
- $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rationals, $\mathbb{R}$ = reals and $\mathbb{C}$ = complex numbers. $\mathbb{N}^+ = \{n \in \mathbb{N} \mid n > 0\}$ and $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$.
- $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.
- In the second half of the book, all Banach algebras are over $\mathbb{C}$.
- The appearance of a terminology in boldface italic font is an indication that its *definition* is to be found nearby.

Sets:

$\emptyset$ (empty set), 1
 $\mathcal{P}(X)$ (power set), 1
 ω (first infinite ordinal), 6
 $\aleph_0$ (first infinite cardinal), 6
 $\aleph_1$ (first uncountable cardinal), 6
 $\prod_U X_i$ (ultraproduct), 19
 $A \setminus B$, 38
 $A \triangle B$ (symmetric difference), 38

Nonstandard analysis:

*X (nonstandard extension), 12
 κ (saturation cardinality), 13
 $V(\mathbb{R})$, $V(^*\mathbb{R})$ (universes), 10,12,16
 $\approx$ (infinitely closed), 23,58,56
 $\mu(x)$ (monad), 23,54,56
 $^\circ x$ (standard part), 23,58
 $\text{Fin}(\cdot)$ (finite part), 23,80
 $|\cdot|$ (internal cardinality), 28
 $\mathbb{T}$ (hyperfinite timeline), 28
 SL^p (integrability), 47
 $\text{ns}(\cdot)$ (nearstandard part), 54
 $\text{st}(\cdot)$ (standard part mapping), 58
 $\hat{X}$ (nonstandard hull), 81
 $\hat{x}$ (element of $\hat{X}$), 81
 $\hat{X}^w$ (general nonst. hull), 121
 $\approx_w$, $\mu_w(\dots)$, $\text{Fin}_w(\cdot)$, 121
 $^*\hat{X}^w$ (weak nonst. hull), 123
 $\text{Fin}_q(\cdot)$, 124
 $\widehat{\mathcal{M}}^\tau$ (tracial nonst. hull), 258
 $\hat{x}^\tau$ (element of $\widehat{\mathcal{M}}^\tau$), 257
 $HL_2(\Omega, \hat{X})$ (hyper-int. func.), 284

Measures:

$\sigma\mathcal{B}$ (σ -algebra), 33
 Leb (Lebesgue measure), 38
 $\bar{\mu}$ (outer measure), 38
 $\underline{\mu}$ (inner measure), 38
 $L(\mu)$ (Loeb measure), 40

a.e. (almost everywhere), 43
 $\alpha(\cdot)$ (intersection number), 48
 $\beta(\cdot)$ (measure number), 48

Topology:

$\mathcal{T}$ (topology), 54
 $\bar{A}$ (topological closure), 54
 $B(x, r)$ (open ball), 55
 $S(x, r)$ (sphere), 55
 X^ϵ (expansion by ϵ), 68
 $\prod_{i \in I} X_i$ (Tychonoff product), 69
 $\text{int}(\cdot)$ (interior of a set), 70
 βX (Stone-Čech comp. of X), 72

Banach space:

$\dim(\cdot)$ (dimension), 77
 $\|\cdot\|$, $\|\cdot\|_X$ (seminorm, norm), 77
 $x + I$ (coset), 79
 X/Y (quotient space), 79
 $\text{dist}(A, B)$ (distance function), 79
 $\text{Lin}(\cdot)$ (linear span), 84
 $\bar{\text{Lin}}(\cdot)$ (closed linear span), 84
 $\|\cdot\|_\infty$ (maximum norm), 87
 $\|\cdot\|_\infty$ (supremum norm), 87
 $\|\cdot\|_\infty$ (essential sup norm), 90
 ℓ_p (sequence space), 87
 c , $c(\mathbb{N})$, c_0 (sequence space), 87
 $\mathbb{M}(\Omega)$, 88
 $C_0(\Omega)$, $C_b(\Omega)$, $C(\Omega)$, 88
 $ba(\mathbb{N})$, $ca(\mathbb{N})$, 89
 $L_p(\mu)$, $L_p(\Omega)$ (Lebesgue space), 89
 $L_p(\Omega, \mathcal{B}, \mu)$ (Lebesgue space), 89
 $W^{k,p}(\Omega)$ (Sobolev space), 90
 $\oplus_p X_i$ (direct sum), 91
 $\text{Ker}(\cdot)$ (kernel), 94
 B_X (open unit ball), 94
 $\bar{B}_X$ (closed unit ball), 94
 S_X (unit sphere), 94
 $\mathcal{B}(X, Y)$ (bdd. lin. operators), 95
 X' (dual space), 95

X'' (bidual), 112
 $\rightarrow_w$ (weak convergence), 123
 $\text{conv}(\cdot)$ (convex hull), 138
 $\overline{\text{conv}}(\cdot)$ (closed convex hull), 138
 $\langle \cdot, \cdot \rangle$ ((pre-)inner product), 148
 $\perp$ (orthogonal), 154
 $X^\perp$ (orthogonal complement), 154
 H_2 (Hardy space), 158
 $\mathcal{B}_c(X, Y)$, $\mathcal{B}_c(X)$, 161

Banach algebra:

$\{\cdot\}'$ (commutant), 163, 252
 f^* (adjoint), 166
 1 (unit element), 182
 $\mathcal{M} \oplus \mathbb{F}$ (unitization), 182
 $\mathcal{B}(\mathbb{F}^n)$ (matrix algebra), 183
 $f \star g$ (convolution), 184
 $\oplus_p \mathcal{M}_i$ (direct sum), 185
 $\mathcal{M}/I$ (quotient algebra), 186
 a^{-1} (inverse), 187
 $\mathcal{M}^{-1}$ (invertible elements), 187
 $\text{GL}(\mathcal{M})$ (general linear group), 189
 $\sigma(\cdot)$, $\sigma_{\mathcal{M}}(\cdot)$ (spectrum), 190
 $\rho(\cdot)$, $\rho_{\mathcal{M}}(\cdot)$ (spectral radius), 191
 e^a (exponential), 198
 $\exp(\mathcal{M})$, 201
 a^* (involution), 208
 $\mathcal{M}_X$, $\mathcal{M}_a$, 210
 $\text{Re}(\mathcal{M})$ (self-adjoint elements), 210
 $\mathcal{M}^+$ (positive elements), 211
 $\text{Proj}(\mathcal{M})$ (projections), 211
 $\mathcal{U}(\mathcal{M})$ (unitary elements), 211
 $\mathcal{U}(\mathcal{M})$ (unitary group), 212
 $\text{Re}(a)$, $\text{Im}(a)$, 214
 $\text{hom}(\mathcal{M})$ (characters), 219
 $\hat{a}$ (Gelfand transform), 220
 $\mathfrak{M}(\mathcal{M})$ (maximal ideals), 220
 $\leq$ (on unital C^* -algebra), 230
 $|a|$ (modulus), 231

$\mathbb{S}(\mathcal{M})$ (states), 235
 (π, H) (GNS representation), 236
 $\oplus_{i \in I} (\pi_i, H_i)$, 238
 $[x, y]$ (commutator), 249
 $\xrightarrow{\text{WOT}}$, 251
 $\xrightarrow{\text{SOT}}$, 251
 $\omega_{\xi, \eta}$, 251
 $\overline{\mathcal{M}^{\text{WOT}}}$, 252
 $\{X\}''$ (bicommutant), 252
 m_f (multiplication operator), 254
 $\int f dp$ (spectral integral), 264
 $p \wedge q$ (on projections), 269
 $p \vee q$ (on projections), 269
 $p \searrow q$ (on projections), 269
 $p \sim q$ (M.-vN. equivalent), 269
 $p \preceq q$ (subordinate), 270
 $a \ltimes b$ (Arens product), 293
 $a \rtimes b$ (Arens product), 293
 ϕ_R , Φ_R , ϕ_L , Φ_L , 300

Miscellaneous:

$\models$ (satisfies), 3
 $(\mathfrak{V}, \varepsilon)$, 11
 χ_S (indicator function), 38
 $\text{graph}(f)$, 75
 $\forall^{\text{fin}}$ (for all finite), 112

Abraham Robinson (1918 - 1974)

List of Theorems

Method of proof: nn = nonstandard ss = standard ns = mixed

zz = proof not included

Abbreviations: nsh = nonstandard hull nst = nonstandard

<i>Alaoglu's Theorem</i>	Thm. 2.16.....p.132	nn
<i>Anderson's Theorem</i>	Thm. 1.13.....p.43	nn
<i>Arens Products on Bidual</i>	Thm. 4.4.....p.293	nn
<i>Baire Category Theorem</i>	Thm. 1.24.....p.70	ns
<i>Banach Contraction Principle</i>	Thm. 1.19.....p.65	ns
<i>Banach-Steinhaus Theorem</i>	Cor. 2.9.....p.105	ss
<i>Bidual of von Neumann Algebra</i>	Thm. 3.26.....p.260	nn
<i>Bidual Theorem</i>	Thm. 2.13.....p.126	nn
<i>Bolzano-Weierstrass Theorem</i>	Thm. 1.5.....p.25	nn
<i>Borel Functional Calculus (nst)</i>	Cor. 3.29.....p.263	nn
<i>Borel Functional Calculus</i>	Cor. 3.28.....p.262	ss
<i>Brouwer's Fixed Point Theorem</i>	Thm. 1.20.....p.65	zz
<i>Cauchy Integral Formula</i>	Thm. 3.3.....p.193	zz
<i>Cauchy's Theorem</i>	Thm. 3.2.....p.193	zz
<i>Closed Graph Theorem</i>	Cor. 2.7.....p.102	ns
<i>Commutative C^*-Algebra</i>	Thm. 3.14.....p.223	ss
<i>Compactness Theorem (first order logic)</i>	Thm. 1.2.....p.4	zz
<i>Continuous Functional Calculus</i>	Cor. 3.18.....p.225	ss
<i>Continuous Inverse</i>	Thm. 3.1.....p.188	nn
<i>Double Sequence</i>	Thm. 1.18.....p.63	nn
<i>Eberlein-Šmulian Theorem</i>	Thm. 2.21.....p.138	ns

<i>Extended Functional Calculus (nst)</i>	Thm. 3.23.....p.248	nn
<i>Extreme Value Theorem</i>	Thm. 1.7.....p.29	nn
<i>Finite Representability</i>	Prop. 2.32.....p.144	nn
<i>Fixed Points in Nsh</i>	Thm. 4.3.....p.290	nn
<i>Gödel's Completeness Theorem</i>	Thm. 1.1.....p.4	zz
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<i>Gelfand-Mazur Theorem</i>	Cor. 3.4.....p.194	ss
<i>General Nsh</i>	Lem. 2.3.....p.122	nn
<i>GNS Construction</i>	Thm. 3.19.....p.237	ns
<i>GNS Theorem</i>	Thm. 3.20.....p.239	ss
<i>Goldstine's Theorem</i>	Thm. 2.17.....p.133	nn
<i>Hahn-Banach Separation Theorem</i>	Thm. 2.11.....p.118	ns
<i>Hahn-Banach Theorem</i>	Thm. 2.9, 2.10.....p.116, 118	ns
<i>Heine-Borel Theorem (general)</i>	Thm. 1.22.....p.69	nn
<i>Helly's Theorem (nst)</i>	Thm. 2.7.....p.112	nn
<i>Helly's Theorem</i>	Cor. 2.10, Thm. 2.8.....p.114, 116	nn
<i>Hilbert Space-valued Integral</i>	Thm. 4.1, 4.2.....p.279, 285	nn
<i>*Isomorphism & Isometry</i>	Thm. 3.15.....p.225	ns
<i>Infinite/Properly Infinite Nsh</i>	Thm. 3.27.....p.271	nn
<i>Intermediate Value Theorem</i>	Thm. 1.6.....p.29	nn
<i>Internal Extension</i>	Prop. 1.1, Lem. 1.2.....p.14, 15	nn
<i>Invariant Subspace Theorem</i>	Thm. 2.31.....p.168	nn
<i>Inverse (nsh)</i>	Cor. 3.11.....p.205	nn
<i>Inverse Mapping Theorem</i>	Cor. 2.6.....p.101	nn
<i>James' Characterization</i>	Thm. 2.23.....p.141	ns
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<i>Open Mapping Theorem</i>	Thm. 2.3.....p.100	nn
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<i>Riesz Functional Calculus</i>	Thm. 3.7.....p.199	zz
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Eduard Helly
(1884 - 1943)

Stefan Banach
(1892 - 1945)

John von Neumann
(1903 - 1957)

Israel M. Gelfand
(1913 - 2009)

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