COMPLEX ANALYSIS

AN INVITATION

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A Concise Introduction to Complex Function Theory

COMPLEX ANALYSIS

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Preface

This textbook is a rigorous introduction to the theory of functions of one complex variable. Much of the material has been used in both graduate and undergraduate courses at Aarhus University. It is our impression that courses in complex function theory these days at many places are postponed to leave space for other topics like point set topology and measure theory. Thus we have felt free to assume that the students as background have some point set topology and calculus of several variables, and that they understand $\epsilon\delta$ arguments. From Chapter X on we even use Lebesgue's dominated and monotone convergence theorems freely. But the proofs are meant for the students and so they are fairly detailed.

We have made an effort to whenever possible to give references to literature that is accessible for the students and/or puts the theory in perspective, both mathematically and otherwise. E.g. to the Chauvenet prize winning paper [Za] and to the fascinating gossip on Bloch's life in [Ca] and its sequel [CF]. Our goal is not to compete with the existing excellent textbooks for historical notes and remarks or for wealth of material. For that we refer the interested reader to, say the monumental work [Bu] and the classic [SG].

We have tried to reach some of the deeper and more interesting results (Picard's theorems, Riemann's mapping theorem, Runge's approximation theorems) rather early, and nevertheless to give the very basic theory an adequate treatment.

Standard notation is enforced throughout. A possible exception is that $B[a,r]$ is the closed ball with center a and radius r in analogy with the notation for a closed interval.

An important part of any course is the set of exercises. We have exercises after each chapter. They are meant to be doable for the students, so we have quite often provided hints about how to proceed.

A couple of times we have succumbed to the temptation of making a digression to an interesting topic, that will not be pursued, e.g. Tauber's theorem, Hadamard's gap theorem and the prime number theorem. We hope that the reader will be irresistibly tempted as well.

The authors welcome correspondence with criticism and suggestions. In particular about literature on the level of students that are about to start their graduate studies in mathematics.

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