PRIMORDIAL COSMOLOGY

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World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Primordial cosmology / by Giovanni Montani ... [et al.].

p. cm.

Includes bibliographical references and index.

ISBN-13: 978-981-4271-00-4 (hardcover : alk. paper)

ISBN-10: 981-4271-00-4 (hardcover : alk. paper)

1. Big bang theory. 2. Singularities (Mathematics) 3. Cosmology--Mathematical models.

I. Montani, Giovanni.

QB991.B54P75 2011

523.1--dc22

2010027346

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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Printed in Singapore.

The authors dedicate this Book to all the people who, like
them, regarded and will regard the understanding of the Early
Universe as the greatest challenge of their lives.

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Preface

Understanding the Origin and Evolution of the Universe is certainly one of the most ambitious and fascinating attempts of the human intelligence. This intellectual and scientific adventure is not only the major approach to give answers to fundamental questions concerning our existence but also this activity provides our intelligence with the special role to be the only (known) tool the Universe has to *investigate* itself, acquiring awareness on the Reality essence.

Despite this ambitious path could discourage from addressing any perspective, the scientific investigation of the Universe has reached surprising and impressive achievements and offers many reliable answers about our origin.

This volume summarizes the most important results of the scientific cosmology, describing the observational knowledge about the Universe evolution and how it allowed the derivation of a theoretical paradigm, able of predictions beyond detected phenomena.

Our analysis is based on a rigorous mathematical characterization of the cosmological topics, finding in the Einstein theory of General Relativity the privileged descriptive physical tool. For the wide and coherent cosmological scenario, this Book is built up as a reference for both students interested to an introductory path, and also for specialists who desire to deepen selected topics.

This Book faces the analysis of many aspects of Modern Cosmology, starting from the presentation of well-grounded assessments on the observed Universe and their theoretical interpretation, up to the discussion of very speculative topics concerning the nature of the *Cosmological singularity*, which are timely for the scientific debate.

The content can be successfully approached by any reader having a

certain familiarity with the concepts and the formalisms at the ground of General Relativity. The presentation is purposely self-consistent when characterizing some canonical topics with a pedagogical perspective, as well as when serving an advanced profile for the subjects requiring a wider background knowledge.

Some peculiar approaches to modern cosmology are treated in their general aspects and included only to provide the reader with a broad vision of the contemporary lines of research, although averting from the core perspective of the Book, and referring to the specific literature for details.

We widely illustrate a series of modern cosmological issues, re-enforcing the idea that the highly symmetric nature of our Universe, as observed at very large scales, is not a primordial notion but results from an evolutionary process of very general initial conditions.

The first part of the Book (Chapters 1 and 2) is devoted to a historical picture of the notion of Universe across the centuries and then addresses the fundamental formalism of General Relativity and differential geometry for the modern approaches to the Early Universe providing the basic notions for the non-specialist reader. The so-called *Physical Cosmology* (Chapters 3-6) is faced in details, introducing the structure and the evolution of the isotropic Universe, according to the *Standard Cosmological Model*. Particular attention is dedicated to the phenomenology of the Universe with respect to the implications for the interpretation of the features implied by the theoretical prescriptions of different models. This aim is also pursued by analyzing the inflationary paradigm as well as a detailed treatment of the density inhomogeneities faced by the perturbation theory to the isotropic Universe and by the paradigm of the *quasi-isotropic solution*.

The part of the Book entitled *Mathematical Cosmology* (Chapters 7-9) gives a wide discussion of the general features of the Universe near the singularity, when the isotropy and homogeneity assumptions are removed. We start with a geometrical characterization of the homogeneous three-dimensional spaces, as arranged in the Bianchi classification, whose dynamics is treated in the framework of General Relativity, also by means of a Hamiltonian formulation. In this respect, our presentation is focused on the study of the chaotic dynamics of the Bianchi type VIII and IX models near the singularity. Here, the discussion clearly separates well-established results from timely reformulations of the problem or aspects yet open to scientific investigation. Finally, we endow this part by a deep analysis of the generic inhomogeneous solution near the cosmological singularity,

derived in analogy to the chaotic homogeneous model and implementing the parametric role played by the space coordinates. The nature of the space-time foam, characterizing the generic Universe asymptotically to the singular point, is outlined toward a proper statistical picture. This study of cosmologies more general than the isotropic Universe is also extended to multidimensional issues, in view of the interest raised through the recent literature.

The last part of the Book is focused on *Quantum Cosmology* (Chapters 10-12) and it touches very timely questions in applying Quantum Gravity proposals to the Universe origin and its primordial evolution. We discuss different approaches to the quantization of the gravitational field, concentrating the investigation on the canonical method, both in the original Wheeler-DeWitt and in the more promising Loop Quantum Gravity approaches. However, different points of view, like the path integral quantization and the generalized Heisenberg non-commutative algebras procedures are also taken into account and implemented on specific models. Our quantum description of the Universe is very general and addresses all the most relevant features, especially in view of the interpretative shortcomings of the different approaches. The recent and outstanding success of Loop Quantum Cosmology in determining the existence of a Big Bounce at the Universe birth is eventually traced with care. Indeed, the quantum nature of the cosmological singularity (its removal or survival) is a central theme and many different issues are contrasted in view of their motivating hypotheses. A part of such Section treats the interpretation of a semiclassical Universe and its tendency to isotropization.

The material presented in the Chapters on quantum cosmology clarifies the motivation for an intense study of the mathematical cosmology, because the general character of the cosmological models in the regime asymptotic to the initial singularity makes them as the most appropriate for the implementation of a quantum theory to the Universe birth. The idea that the classical Universe comes out of the Planck epoch via a semiclassical limit, say when its volume expectation value is large enough, implies that we must have a clear understanding of such general dynamics already at a classical level. The possibility to link the anisotropic and inhomogeneous cosmologies to the isotropic Universe, underlying the Standard Cosmological model, can be recognized in the inflation scenario. In fact, the vacuum energy responsible for the de Sitter phase of the inflating Universe is a strong isotropic term, able to stretch the inhomogeneities at scales much

larger than the physical horizon and to suppress anisotropic features.

This scenario prescribes that the Universe is born from a singularity-free generic inhomogeneous model; when its volume is probabilistically large enough, it transits to a quasi-classical dynamics, and then it is reconciled to the Standard Cosmological Model by the inflationary process. Such a point of view constitutes the leading perspective suggested and partially demonstrated by this Book.

We would like to express our gratitude to Dr Massimiliano Lattanzi for his precious contribution to the part of this Book devoted to *Physical Cosmology*, especially in view of the effort made to link the theoretical framework to the present knowledge of the observed Universe.

Dr Nakia Carlevaro and Dr Francesco Cianfrani are thanked for their help in writing Sec. 3.6 on the Lemaître-Tolmann-Bondi model. F.C. is also thanked for his comments and suggestions on the part of the Book concerning *Quantum Cosmology*.

We would like to acknowledge Dr Simone Speziale for his valuable suggestions on that part in which we review the main features of the Loop Quantum Gravity theory.

Finally, a special thought of G.M. is devoted to the memory of Prof. Kensuke Yoshida, who appreciably encouraged him to write the review article¹, from which the project of this Book arose.

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¹Montani, G., Battisti, M.V., Benini, R. and Imponente, G.(2008). Classical and Quantum Features of the Mixmaster Singularity, *Int. J. Mod. Phys. A* **23**, pp. 2353 - 2503, doi:10.1142/S0217751X08040275.

$\kappa = 8\pi G$	Einstein constant
$\hbar = c = 1$	natural units
$(+, -, -, -)$	metric signature
g_{ij}	metric tensor
$h_{\alpha\beta}$	spatial metric tensor
ds^2	space-time line element
dl^2	spatial line element
$\mathcal{M}$	space-time manifold
Σ	space-like surface
S	action
$\mathcal{L}$	Lagrangian density
$\mathcal{H}$	Hamiltonian density
H	Hubble parameter
ρ	energy density
P	pressure
p	particle momentum
$i, j, k, \dots$	space-time indices
$\alpha, \beta, \gamma, \dots$	spatial indices
$I, J, K, \dots$	internal 4-d indices
$a, b, c, \dots$	internal 3-d indices

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