PRIMORDIAL COSMOLOGY

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The authors dedicate this Book to all the people who, like them, regarded and will regard the understanding of the Early Universe as the greatest challenge of their lives.

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Preface

Understanding the Origin and Evolution of the Universe is certainly one of the most ambitious and fascinating attempts of the human intelligence. This intellectual and scientific adventure is not only the major approach to give answers to fundamental questions concerning our existence but also this activity provides our intelligence with the special role to be the only (known) tool the Universe has to *investigate* itself, acquiring awareness on the Reality essence.

Despite this ambitious path could discourage from addressing any perspective, the scientific investigation of the Universe has reached surprising and impressive achievements and offers many reliable answers about our origin.

This volume summarizes the most important results of the scientific cosmology, describing the observational knowledge about the Universe evolution and how it allowed the derivation of a theoretical paradigm, able of predictions beyond detected phenomena.

Our analysis is based on a rigorous mathematical characterization of the cosmological topics, finding in the Einstein theory of General Relativity the privileged descriptive physical tool. For the wide and coherent cosmological scenario, this Book is built up as a reference for both students interested to an introductory path, and also for specialists who desire to deepen selected topics.

This Book faces the analysis of many aspects of Modern Cosmology, starting from the presentation of well-grounded assessments on the observed Universe and their theoretical interpretation, up to the discussion of very speculative topics concerning the nature of the *Cosmological singularity*, which are timely for the scientific debate.

The content can be successfully approached by any reader having a

certain familiarity with the concepts and the formalisms at the ground of General Relativity. The presentation is purposely self-consistent when characterizing some canonical topics with a pedagogical perspective, as well as when serving an advanced profile for the subjects requiring a wider background knowledge.

Some peculiar approaches to modern cosmology are treated in their general aspects and included only to provide the reader with a broad vision of the contemporary lines of research, although averting from the core perspective of the Book, and referring to the specific literature for details.

We widely illustrate a series of modern cosmological issues, re-enforcing the idea that the highly symmetric nature of our Universe, as observed at very large scales, is not a primordial notion but results from an evolutionary process of very general initial conditions.

The first part of the Book (Chapters 1 and 2) is devoted to a historical picture of the notion of Universe across the centuries and then addresses the fundamental formalism of General Relativity and differential geometry for the modern approaches to the Early Universe providing the basic notions for the non-specialist reader. The so-called *Physical Cosmology* (Chapters 3-6) is faced in details, introducing the structure and the evolution of the isotropic Universe, according to the *Standard Cosmological Model*. Particular attention is dedicated to the phenomenology of the Universe with respect to the implications for the interpretation of the features implied by the theoretical prescriptions of different models. This aim is also pursued by analyzing the inflationary paradigm as well as a detailed treatment of the density inhomogeneities faced by the perturbation theory to the isotropic Universe and by the paradigm of the *quasi-isotropic solution*.

The part of the Book entitled *Mathematical Cosmology* (Chapters 7-9) gives a wide discussion of the general features of the Universe near the singularity, when the isotropy and homogeneity assumptions are removed. We start with a geometrical characterization of the homogeneous three-dimensional spaces, as arranged in the Bianchi classification, whose dynamics is treated in the framework of General Relativity, also by means of a Hamiltonian formulation. In this respect, our presentation is focused on the study of the chaotic dynamics of the Bianchi type VIII and IX models near the singularity. Here, the discussion clearly separates well-established results from timely reformulations of the problem or aspects yet open to scientific investigation. Finally, we endow this part by a deep analysis of the generic inhomogeneous solution near the cosmological singularity,

derived in analogy to the chaotic homogeneous model and implementing the parametric role played by the space coordinates. The nature of the space-time foam, characterizing the generic Universe asymptotically to the singular point, is outlined toward a proper statistical picture. This study of cosmologies more general than the isotropic Universe is also extended to multidimensional issues, in view of the interest raised through the recent literature.

The last part of the Book is focused on *Quantum Cosmology* (Chapters 10-12) and it touches very timely questions in applying Quantum Gravity proposals to the Universe origin and its primordial evolution. We discuss different approaches to the quantization of the gravitational field, concentrating the investigation on the canonical method, both in the original Wheeler-DeWitt and in the more promising Loop Quantum Gravity approaches. However, different points of view, like the path integral quantization and the generalized Heisenberg non-commutative algebras procedures are also taken into account and implemented on specific models. Our quantum description of the Universe is very general and addresses all the most relevant features, especially in view of the interpretative shortcomings of the different approaches. The recent and outstanding success of Loop Quantum Cosmology in determining the existence of a Big Bounce at the Universe birth is eventually traced with care. Indeed, the quantum nature of the cosmological singularity (its removal or survival) is a central theme and many different issues are contrasted in view of their motivating hypotheses. A part of such Section treats the interpretation of a semiclassical Universe and its tendency to isotropization.

The material presented in the Chapters on quantum cosmology clarifies the motivation for an intense study of the mathematical cosmology, because the general character of the cosmological models in the regime asymptotic to the initial singularity makes them as the most appropriate for the implementation of a quantum theory to the Universe birth. The idea that the classical Universe comes out of the Planck epoch via a semiclassical limit, say when its volume expectation value is large enough, implies that we must have a clear understanding of such general dynamics already at a classical level. The possibility to link the anisotropic and inhomogeneous cosmologies to the isotropic Universe, underlying the Standard Cosmological model, can be recognized in the inflation scenario. In fact, the vacuum energy responsible for the de Sitter phase of the inflating Universe is a strong isotropic term, able to stretch the inhomogeneities at scales much

larger than the physical horizon and to suppress anisotropic features.

This scenario prescribes that the Universe is born from a singularity-free generic inhomogeneous model; when its volume is probabilistically large enough, it transits to a quasi-classical dynamics, and then it is reconciled to the Standard Cosmological Model by the inflationary process. Such a point of view constitutes the leading perspective suggested and partially demonstrated by this Book.

We would like to express our gratitude to Dr Massimiliano Lattanzi for his precious contribution to the part of this Book devoted to *Physical Cosmology*, especially in view of the effort made to link the theoretical framework to the present knowledge of the observed Universe.

Dr Nakia Carlevaro and Dr Francesco Cianfrani are thanked for their help in writing Sec. 3.6 on the Lemaître-Tolmann-Bondi model. F.C. is also thanked for his comments and suggestions on the part of the Book concerning *Quantum Cosmology*.

We would like to acknowledge Dr Simone Speziale for his valuable suggestions on that part in which we review the main features of the Loop Quantum Gravity theory.

Finally, a special thought of G.M. is devoted to the memory of Prof. Kensuke Yoshida, who appreciably encouraged him to write the review article¹, from which the project of this Book arose.

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$\kappa = 8\pi G$	Einstein constant
$\hbar = c = 1$	natural units
$(+, -, -, -)$	metric signature
g_{ij}	metric tensor
$h_{\alpha\beta}$	spatial metric tensor
ds^2	space-time line element
dl^2	spatial line element
$\mathcal{M}$	space-time manifold
Σ	space-like surface
S	action
$\mathcal{L}$	Lagrangian density
$\mathcal{H}$	Hamiltonian density
H	Hubble parameter
ρ	energy density
P	pressure
p	particle momentum
$i, j, k, \dots$	space-time indices
$\alpha, \beta, \gamma, \dots$	spatial indices
$I, J, K, \dots$	internal 4-d indices
$a, b, c, \dots$	internal 3-d indices

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Contents

<i>Preface</i>	vii
<i>List of Figures</i>	xxiii
Historical and Basic Notions	1
1. Historical Picture	3
1.1 The Concept of Universe Through the Centuries	3
1.1.1 The ancient cultures	3
1.1.2 Ancient Greek and the Mediterranean	5
1.1.3 The Hellenistic era	9
1.1.4 The philosophical point of view in the Old Age . .	11
1.1.5 The Middle Age dogmas	11
1.1.6 The Renaissance revolution	15
1.1.7 The Scientific Revolution	16
1.1.8 The Enlightenment Era	23
1.2 The XIX Century Knowledge	25
1.2.1 Geometrical formalisms	25
1.2.2 Difficulties for the birth of a real cosmology: Ol- bers' paradox	27
1.2.3 Luigi Bianchi and the developments of differential geometry	28
1.2.4 Einstein vision of space-time	28
1.3 Birth of Scientific Cosmology	30
1.3.1 Einstein proposal of a static Universe	31

1.3.2	Galaxies and their expansion: The Hubble’s discovery	32
1.4	The Genesis of the Hot Big Bang Model	34
1.4.1	Recent developments	37
1.4.2	Discovery of the acceleration	40
1.4.3	Generic nature of the cosmological singularity: The Cambridge and the Landau School	41
1.4.4	The inflationary paradigm	43
1.4.5	The idea of non-singular cosmology: The cyclic Universe and the Big Bounce	44
1.5	Guidelines to the Literature	46
2.	Fundamental Tools	51
2.1	Einstein Equations	52
2.2	Matter Fields	54
2.2.1	Perfect fluid	54
2.2.2	Scalar field	56
2.2.3	Electromagnetic field	58
2.2.4	Yang-Mills fields and Θ -sector	60
2.3	Hamiltonian Formulation of the Dynamics	63
2.3.1	Canonical General Relativity	64
2.3.2	Hamilton-Jacobi equations for gravitational field	70
2.3.3	The ADM reduction of the dynamics	71
2.4	Synchronous Reference System	73
2.5	Tetradic Formalism	74
2.6	Gauge-like Formulation of GR	77
2.6.1	Lagrangian formulation	77
2.6.2	Hamiltonian formulation	80
2.6.3	On the gauge group of GR	84
2.7	Singularity Theorems	85
2.7.1	Definition of a space-time singularity	85
2.7.2	Fluid kinematics	86
2.7.3	The Raychaudhuri equation	88
2.7.4	Singularity Theorems	90
2.8	Guidelines to the Literature	92

Physical Cosmology	93
3. The Structure and Dynamics of the Isotropic Universe	95
3.1 The RW Geometry	96
3.1.1 Definition of isotropy	97
3.1.2 Kinematics of the isotropic Universe	98
3.1.3 The particle motion	99
3.1.4 The Hubble law	100
3.1.5 The Hubble length and the cosmological horizon .	103
3.1.6 Kinetic theory and thermodynamics in the expanding Universe: The hot Big Bang	106
3.2 The FRW Cosmology	112
3.2.1 Field equations for the isotropic Universe	112
3.2.2 Asymptotic solution toward the Big Bang	114
3.2.3 The de Sitter Solution	118
3.2.4 Hamiltonian dynamics of the isotropic Universe .	119
3.3 Dissipative Cosmologies	122
3.3.1 Bulk viscosity	122
3.3.2 Matter creation in the expanding Universe	124
3.4 Inhomogeneous Fluctuations in the Universe	127
3.4.1 The meaning of cosmological perturbations	127
3.4.2 The Jeans length in a static uniform fluid	131
3.4.3 The Jeans length in an expanding Universe	132
3.5 General Relativistic Perturbation Theory	136
3.5.1 Perturbed Einstein equations	137
3.5.2 Scalar-vector-tensor decomposition and Fourier expansion	140
3.5.3 Perturbed conservation equations	143
3.5.4 Gauge modes	144
3.5.5 Evolution of scalar modes	147
3.5.6 Adiabatic and isocurvature perturbations	149
3.5.7 Imperfect fluids	151
3.5.8 Kinetic theory	154
3.6 The Lemaître-Tolmann-Bondi Spherical Solution	159
3.7 Guidelines to the Literature	162
4. Features of the Observed Universe	165
4.1 Current Status: The Concordance Model	166

4.2	The Large-Scale Structure	168
4.2.1	Deviations from homogeneity	169
4.2.2	Dark matter	170
4.2.3	The power spectrum of density fluctuations	172
4.3	The Acceleration of the Universe	175
4.4	The Cosmic Microwave Background	176
4.4.1	Sources of anisotropy	179
4.4.2	The power spectrum of CMB anisotropies	180
4.4.3	Acoustic oscillations	183
4.4.4	Effect of the cosmological parameters	187
4.5	Guidelines to the Literature	191
5.	The Theory of Inflation	193
5.1	The Shortcomings of the Standard Cosmology	194
5.1.1	The horizon and flatness paradoxes	195
5.1.2	The entropy problem and the unwanted relics paradox	197
5.2	The Inflationary Paradigm	200
5.2.1	Spontaneous symmetry breaking and the Higgs phenomenon	200
5.3	Presence of a Self-interacting Scalar Field	205
5.3.1	Coupling of the scalar field with the thermal bath	211
5.4	Inflationary Dynamics	213
5.4.1	Slow-rolling phase	214
5.4.2	The reheating phase	215
5.5	Solution to the Shortcomings of the Standard Cosmology	221
5.5.1	Solution to the horizon and flatness paradoxes	221
5.5.2	Solution to the entropy problem and to the unwanted relics paradox	224
5.6	General Features	225
5.6.1	Slow-rolling phase	225
5.6.2	Reheating phase	226
5.6.3	The Coleman-Weinberg model	227
5.6.4	Genesis of the seeds for structure formation	228
5.7	Possible Explanations for the Present Acceleration of the Universe	233
5.7.1	Dark energy	234
5.7.2	Modified gravity theory	236
5.8	Guidelines to the Literature	239

6.	Inhomogeneous Quasi-isotropic Cosmologies	241
6.1	Quasi-Isotropic Solution	242
6.2	The Presence of Ultrarelativistic Matter	242
6.3	The Role of a Massless Scalar Field	246
6.4	The Role of an Electromagnetic Field	252
6.5	Quasi-Isotropic Inflation	257
6.5.1	Geometry, matter and scalar field equations . . .	258
6.5.2	Inflationary dynamics	259
6.5.3	Physical considerations	262
6.6	Quasi-Isotropic Viscous Solution	264
6.6.1	Form of the energy density	265
6.6.2	Comments on the adopted paradigm	266
6.6.3	Solutions of the 00-Einstein and hydrodynamical equations	270
6.6.4	The velocity and the three-metric	272
6.7	Guidelines to the Literature	274

Mathematical Cosmology 277

7.	Homogeneous Universes	279
7.1	Homogeneous Cosmological Models	280
7.1.1	Definition of homogeneity	280
7.1.2	Application to Cosmology	283
7.1.3	Bianchi Classification and Line Element	286
7.2	Kasner Solution	288
7.2.1	The role of matter	291
7.3	The Dynamics of the Bianchi Models	292
7.3.1	Bianchi type II: The concept of Kasner epoch . .	294
7.3.2	Bianchi type VII: The concept of Kasner era . . .	297
7.4	Bianchi Types VIII and IX Models	298
7.4.1	The oscillatory regime	298
7.4.2	Stochastic properties and the Gaussian distribution	301
7.4.3	Small oscillations	304
7.5	Dynamical Systems Approach	308
7.5.1	Equations for orthogonal Bianchi class A models .	310
7.5.2	The Bianchi I model and the Kasner circle	312
7.5.3	The Bianchi II model in vacuum	315

7.5.4	The Bianchi IX model and the Mixmaster attractor theorem	317
7.6	Multidimensional Homogeneous Universes	320
7.6.1	On the non-diagonal cases	323
7.7	Guidelines to the Literature	325
8.	Hamiltonian Formulation of the Mixmaster	327
8.1	Hamiltonian Formulation of the Dynamics	328
8.2	The Mixmaster Model in the Misner Variables	331
8.2.1	Metric reparametrization	332
8.2.2	Kasner solution	333
8.2.3	Lagrangian formulation	334
8.2.4	Reduced ADM Hamiltonian	334
8.2.5	Mixmaster dynamics	335
8.3	Misner-Chitré Like Variables	339
8.3.1	The Jacobi metric and the billiard representation	341
8.3.2	Some remarks on the billiard representation	344
8.4	The Invariant Liouville Measure	344
8.5	Invariant Lyapunov Exponent	346
8.6	Chaos Covariance	348
8.6.1	Shortcomings of Lyapunov exponents	349
8.6.2	On the occurrence of fractal basin	351
8.7	Cosmological Chaos as a Dimensional and Matter Dependent Phenomenon	353
8.7.1	The role of a scalar field	353
8.7.2	The role of a vector field	355
8.8	Isotropization Mechanism	356
8.9	Guidelines to the Literature	359
9.	The Generic Cosmological Solution Near the Singularity	361
9.1	Inhomogeneous Perturbations of Bianchi IX	362
9.2	Formulation of the Generic Cosmological Problem	366
9.2.1	The Generalized Kasner solution	367
9.2.2	Inhomogeneous BKL solution	368
9.2.3	Rotation of the Kasner axes	372
9.3	The Fragmentation Process	373
9.3.1	Physical meaning of the BKL conjecture	375
9.4	The Generic Cosmological Solution in Misner Variables	375

9.5	Hamiltonian Formulation in a General Framework	378
9.5.1	Local dynamics	380
9.5.2	Dynamics of inhomogeneities	381
9.6	The Generic Cosmological Problem in the Iwasawa Variables	383
9.6.1	Asymptotic freezing of the Iwasawa variables . . .	385
9.6.2	Cosmological billiards	386
9.7	Multidimensional Oscillatory Regime	388
9.7.1	Dilatons, p -forms and Kac-Moody algebras	390
9.8	Properties of the BKL Map	391
9.8.1	Parametrization in a generic number of dimensions	392
9.8.2	Ordering properties	394
9.8.3	Properties of the BKL map in the v space	397
9.9	Guidelines to the Literature	398

Quantum Cosmology 401

10.	Standard Quantum Cosmology	403
10.1	Quantum Geometrodynamics	405
10.1.1	The Wheeler-DeWitt Theory	405
10.1.2	Relation with the Path Integral Quantization . . .	410
10.2	The Problem of Time	412
10.2.1	Time before quantization	413
10.2.2	Time after quantization	418
10.2.3	Timeless physics	419
10.3	What is Quantum Cosmology?	421
10.3.1	Minisuperspace models	422
10.3.2	Interpretation of the theory	424
10.3.3	Quantum singularity avoidance	425
10.4	Path Integral in the Minisuperspace	427
10.5	Scalar Field as Relational Time	430
10.6	Interpretation of the Wave Function of the Universe . . .	433
10.6.1	The semiclassical approximation	434
10.6.2	An example: A quantum mechanism for the isotropization of the Universe	437
10.7	Boundary Conditions	441
10.7.1	No-boundary proposal	443
10.7.2	Tunneling proposal	445

10.7.3	Comparison between the two approaches	446
10.8	Quantization of the FRW Model Filled with a Scalar Field	447
10.9	The Poincaré Half Plane	451
10.10	Quantum Dynamics of the Taub Universe	453
10.10.1	Classical framework	453
10.10.2	Quantum framework	454
10.11	Quantization of the Mixmaster in the Misner Picture	457
10.12	The Quantum Mixmaster in the Poincaré Half Plane	459
10.12.1	Continuity equation and the Liouville theorem	460
10.12.2	Schrödinger dynamics	462
10.12.3	Eigenfunctions and the vacuum state	464
10.12.4	Properties of the spectrum	466
10.13	Guidelines to the Literature	469
11.	Generalized Approaches to Quantum Mechanics	473
11.1	The Algebraic Approach	474
11.1.1	Basic elements	475
11.1.2	GNS construction and Fell theorem	478
11.2	Polymer Quantum Mechanics	480
11.2.1	From Schrödinger to polymer representation	481
11.2.2	Kinematics	483
11.2.3	Dynamics	485
11.3	On the Existence of a Fundamental Scale	487
11.4	String Theory and Generalized Uncertainty Principle	488
11.5	Heisenberg Algebras in Non-Commutative Snyder Space-Time	491
11.6	Quantum Mechanics in the GUP Framework	494
11.7	Guidelines to the Literature	498
12.	Modern Quantum Cosmology	499
12.1	Loop Quantum Gravity	500
12.1.1	Kinematics	502
12.1.2	Implementation of the constraints	506
12.1.3	Quantum constraints algebra	510
12.2	Loop Quantum Cosmology	511
12.2.1	Kinematics	511
12.2.2	Quantum dynamics and Big Bounce	516
12.2.3	Effective classical dynamics	519

12.3	Mixmaster Universe in LQC	521
12.3.1	Loop quantum Bianchi IX	522
12.3.2	Effective dynamics	526
12.4	Triangulated Loop Quantum Cosmology	528
12.4.1	The triangulated model	528
12.4.2	Isotropic sector: FRW	531
12.4.3	Anisotropic sector: Bianchi IX	532
12.4.4	Full dipole model	533
12.4.5	Quantization of the model	535
12.5	Snyder-Deformed Quantum Cosmology	536
12.6	GUP and Polymer Quantum Cosmology: The Taub Universe	540
12.6.1	Deformed classical dynamics	541
12.6.2	Deformed quantum dynamics	542
12.7	Mixmaster Universe in the GUP Approach	546
12.8	Guidelines to the Literature	550

<i>Bibliography</i>	553
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<i>Index</i>	585
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List of Figures

2.1	Construction of the lapse function and of the shift vector . . .	65
4.1	Luminosity distance vs. redshift	175
4.2	CMB frequency spectrum	176
4.3	Microwave sky	177
4.4	Acoustic oscillations	183
4.5	CMB anisotropy spectrum from WMAP7	189
5.1	Higgs potential	201
5.2	Old inflation potential	207
5.3	Slow rolling potential	208
5.4	Temperature-dependent Higgs potential	210
5.5	Behavior of the energy density during reheating	217
7.1	Kasner indices in terms of the parameter $1/u$	287
7.2	The Kasner Circle	310
7.3	Dynamics of Ω and Σ_a in the Bianchi I model with matter . .	312
7.4	Bianchi II transitions in Σ plane	313
7.5	Bianchi IX evolution	314
7.6	Region where the 5-d BKL mechanism breaks down	320
8.1	Equipotential lines of the Bianchi type VIII model	332
8.2	Equipotential lines of the Bianchi type IX model	333
8.3	Reduced configurational space $\Gamma_Q(\xi, \theta)$	338
8.4	Fractal structure in Mixmaster dynamics	348
9.1	Domain of validity for the 4-d Kasner parametrization.	391
9.2	3-d representation of p_i of the 4-d Kasner model.	392

10.1 Ground state of an isotropizing Universe 437

10.2 The Hartle-Hawking instanton 440

10.3 Triangular domain $\Pi_Q(u, v)$ in the Poincaré plane 448

10.4 Dynamics of the Taub Universe in the (τ, x) -plane 450

10.5 WDW wave packet for the Taub Universe 452

10.6 $\Gamma_n(\alpha)$ for three different values of the parameter k_n 455

10.7 Approximate domain for the quantum dynamics of Bianchi IX 460

10.8 Wave function of the ground-state of the Mixmaster 462

11.1 Region allowed by the generalized uncertainty principle 486

12.1 Effective LQC vs classical dynamics for the FRW Universe . . 517

12.2 GUP wave-function for the Taub Universe 540

12.3 Polymer wave packet $|\Psi(x, \tau)|$ of the Taub Universe. 541

12.4 Velocity of the wall of Bianchi IX in the GUP framework . . . 545