Introduction to Quantum Groups

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Introduction to *Quantum Groups*

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To our parents

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Preface

The importance of the theory of Lie groups and algebras for theoretical physics can hardly be disputed. Its applications in most branches of modern physics have proved to be extremely fruitful not only for the solution of already existing problems but also as a guide for the formulation and development of essentially new ideas and approaches for the description of physical phenomena. It turns out, however, that the present level of theoretical problems requires more general algebraic objects called *quantum groups* which contain Lie groups and Lie algebras as particular and, in a sense, degenerate cases. The mathematical theory of these new objects arose as an abstraction from constructions developed in the frame of the inverse scattering method of solution of quantum integrable models. This origin implied, from the very beginning, the importance of this mathematical theory for physics. On the other hand, the theory has proved to be so rich and powerful that it seems natural to apply it to different problems far beyond the original area, generalizing such basic physical notions as space-time symmetries, gauge symmetries, different types of dynamical symmetries and so on.

At present the theory being highly developed, is still far from full completeness. Its mathematical foundation are summarized in a number of books [171, 64, 61, 137, 174]. But many results, especially concerning physical appli-

cations, are scattered in a variety of original papers and reviews, often rather difficult for a first reading.

In this book we have aimed at giving an introduction to some of the principal ideas of the quantum group theory accompanied by examples of its different applications, assuming that the reader is familiar with the theory of Lie groups, Lie algebras and quantum mechanics. A certain acquaintance with some basic facts from the theory of supergroups and quantum field theory is also assumed. Actually, the latter is important only for a better understanding of some applications of the new mathematics. Though the most natural applications of quantum groups are in the theory of integrable systems, we have not concentrated on this subject. The main reason is that we have intended to show a variety and diversity of possible applications of quantum groups, including the theory of elementary particles and field theory. Besides, the inverse scattering method is a very extensive subject which cannot be reduced only to quantum groups. Its self-contained exposition would require an essential increase of the volume of the book and would lead us away from the main topic.

By now quantum groups and, in a more general sense, non-commutative geometry have become an immense and versatile subject. We have selected certain topics for discussion and we are well aware of the fact that many important topics have been either left out or are only mentioned briefly. We hope that partially this is compensated by references in our book to the original papers and appropriate reviews. Also we have tried to use a minimum of mathematical tools. Thus proofs of a number of theorems and details of applications are either briefly sketched or omitted, adequate references being given to enable the interested reader to study the subject in full completeness.

In writing this book we have benefited from discussions on various topics with many of our colleagues, in particular with Peter Prešnajder. We thank all of them for useful discussions and for their advice. Especially, it is a pleasure to express our gratitude to Peter Kulish and to acknowledge stimulating discussions, suggestions and criticism. We are deeply indebted for all what he has offered to us. We are also grateful to Wen-Feng Chen for making many improvements in the manuscript. A.D. is much grateful to José Helayel-Neto for constant support and to the Brazilian Research Council and INTAS for financial support.

Helsinki, Rio de Janeiro
March 1996

M. Chaichian, A. Demichev

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