Innovations in Insurance, Risk- and Asset Management

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Chapter 16

Forward versus Spot Price Modeling

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It is possible to base an equity derivatives pricing model on a stochastic driving process for the share price (spot), or for the equity forward. While the former is the classical approach pioneered by Black and Scholes [9], the latter approach separates the modeling of exogenous random price fluctuations from the cost-of-carry modeling of the stock, probably the first and most prominent example of this technique being the paper by Black [8]. While the Black–Scholes spot price approach and Black’s forward approach are equivalent, the present note demonstrates that the introduction of local volatility and/ or level-dependent default intensity into the stochastic driving process destroys this equivalence, if applied carelessly. The advantages and disadvantages of both approaches are discussed.

Keywords: equity forward, derivative pricing, dividend modeling, credit-equity modeling, CEV model, JDCEV model.

1. Introduction

Today’s equity derivative pricing algorithms are based on stochastic modeling of the underlying equity. The pioneering paper by Black and Scholes [9] paved the road for dozens of different models with varying complexity, which all have in common that the underlying share price process is modeled exogenously by some more or less complex stochastic process. The price of a derivative is then given as the expected value of the derivative’s payoff function with respect to a so-called risk-neutral probability measure. Due to arbitrage constraints the drift of the share price process under the risk-neutral pricing measure is determined, in the sense that the stochastic process which describes the discounted wealth from holding a portfolio that is long one share becomes a (local) martingale.¹ The incorporation

¹This is roughly the statement of the so-called 1st fundamental theorem of asset pricing, cf. [13], [14] for detailed background.

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of a deterministic interest rate, a continuous rate accounting for proceeds from holding the stock (e.g. due to dividends and/ or the possibility for stock lending via repos²), or even discrete cash dividends into the stock price model in general is a non-trivial, or at least inconvenient, task that must be executed in accordance with these arbitrage constraints. Not much later than the invention of the pioneering Black–Scholes model in 1973, in 1976 Black [8] already proposed a trick to circumvent this difficult task by modeling the equity forward³ process exogenously (instead of the share price). While the spot price S_t of an equity is the price at which one share can be bought or sold at time t , the equity forward $F(t, T)$ is defined as the market's expectation at time t about the value of the spot at some fixed future time $T \geq t$. The difference between the share price (spot) and its forward is the cost-of-carry between t and T , which consists precisely of the components mentioned before: interest rates, repo rates, and dividends. Consequently, the task of modeling the forward process $\{F(t, T)\}_{t \in [0, T]}$ exogenously is a task that is free from these considerations,⁴ and after having modeled the forward process exogeneously as an arbitrary (local) martingale, one may include interest rate, repo, and dividend considerations in a separate step.

For the sake of a clear presentation, throughout the present paper we assume that there is a deterministic, risk-free short rate $r(\cdot)$ used for discounting cash flows, and a deterministic, continuous yield $\delta(\cdot)$ accounting for proceeds from stock possession, so that at time t the relation between the share price S_t and the equity forward $F(t, T)$ with maturity $T > t$ is given by

$$S_t = e^{-\int_t^T (r(s) - \delta(s)) ds} F(t, T), \quad 0 \leq t \leq T. \quad (1)$$

In particular, for the sake of simplicity the model abstracts from the fact that in reality dividends are paid at discrete time points, but we discuss the possibility for the inclusion of discrete cash dividends in Section 5. Furthermore, the inclusion of stochastic interest rates is no problem on a generic level, but only disturbs from the main message we like to convey, which is the sole reason why we assume interest rates to be deterministic

²By “repo” we mean a repurchase agreement.

³Black [8] considers a commodity as underlying and distinguishes “spot” and “futures price”, which in our equity setting and today's terminology corresponds to “spot” and “equity forward”.

⁴Except for the starting value $F(0, T)$, which depends on the current spot price S_0 as well as the market's expected cost-of-carry between $[0, T]$, see Formula (1) for $t = 0$.

throughout. Equation (1) shows that one of the two stochastic quantities S_t and $F(t, T)$ is obtained from the other by multiplying it with a deterministic process under our assumptions. When it comes to stochastic modeling, the only constraint from arbitrage pricing theory is that the process $\{F(t, T)\}_{t \in [0, T]}$ is a (local) martingale. For instance, if $\{F(t, T)\}_{t \in [0, T]}$ is modeled as a driftless diffusion, this implies that $\{S_t\}_{t \in [0, T]}$ is a diffusion with drift $r(\cdot) - \delta(\cdot)$. Consequently, one would guess that it makes no difference whether one models the spot process $\{S_t\}_{t \in [0, T]}$ exogenously and derives the forward process $\{F(t, T)\}_{t \in [0, T]}$ from it, or vice versa. Indeed, the classical Black-Scholes spot model defines $\{S_t\}_{t \in [0, T]}$ as a geometric Brownian motion, which is equivalent to the Black forward model that defines the forward process $\{F(t, T)\}_{t \in [0, T]}$ as a geometric Brownian motion, with the same volatility but different drift. However, the present article points out that this equivalence hinges upon the assumption of constant volatility. When the driving geometric Brownian motion is replaced by some more general diffusion process with state-dependent local volatility, the parameters $r(\cdot)$ and $\delta(\cdot)$ must be incorporated in a deliberate way into the local volatility term in order for both approaches to be equivalent, see the following example.

Example 1.1 (Drift effect on local volatility). Assume that the forward process $\{F(t, T)\}_{t \in [0, T]}$ follows a local volatility model of the form

$$dF(t, T) = F(t, T) \sigma_F(F(t, T), t) dW_t, \quad (2)$$

where $\{W_t\}$ denotes a standard Brownian motion. According to (1), Itô calculus implies that $\{S_t\}_{t \in [0, T]}$ follows the diffusion

$$dS_t = S_t \left([r(t) - \delta(t)] dt + \sigma_S(S_t, t) dW_t \right), \quad (3)$$

where the spot price local volatility σ_S is related to the forward local volatility σ_F via

$$\sigma_S(S_t, t) := \sigma_F \left(S_t e^{\int_t^T r(s) - \delta(s) ds}, t \right) = \sigma_F(F(t, T), t). \quad (4)$$

Consequently, apart from the differing drift terms in (2) and (3), that result from no arbitrage considerations, also the forward local volatility functions σ_F and σ_S must differ by the discount factor term in the middle expression in (4) in order for (2) and (3) to become equivalent. If, instead of (3), we model the share price dynamics as

$$dS_t = S_t \left([r(t) - \delta(t)] dt + \sigma_F(S_t, t) dW_t \right), \quad (5)$$

i.e. if we use the same driving process as in the forward model only with different drift, we end up with a spot price model that in general is not equivalent to the forward model (2), even though it is also a share price model that is consistent with arbitrage pricing theory.

In Example 1.1 we purposely distinguish the spot local volatility $\sigma_S(.,.)$ and the forward local volatility $\sigma_F(.,.)$, because they are different in general. In practical modeling, in a first (purely stochastic, non-economic) step, one typically decides for some diffusion model with drift $\mu(.)$ and a parametric family of local volatility functions $\sigma(.,.)$, still without economic meaning. The economic interpretation is attached to the model in a second step, when one decides whether the diffusion represents the share price (in which case one sets $\mu = r - \delta$) or the forward (in which case one sets $\mu(t) \equiv 0$). However, it is unusual to incorporate the parameters $r(.)$ and $\delta(.)$ into the local volatility model $\sigma(.,.)$. The present article's purpose is to point out that the decision between "spot modeling" and "forward modeling" can have economic effects that one should keep in mind. Concretely, we investigate the difference between the two different spot price dynamics (3) and (5) of Example 1.1. Furthermore, in order to be even more general, additional to the local volatility part we also include a state-dependent default intensity into the model. For the state-dependence of the latter default intensity there are similar constraints for the equivalence of spot and forward model as in the case of the local volatility.

The remainder of the article is organized as follows. Section 2 introduces both the spot model and the forward model based on a general defaultable Markov diffusion as stochastic driving process. Section 3 provides an elaborate example for how different both modeling approaches can be, when a so-called CEV driving diffusion is chosen. However, the example is still rather theoretic with the sole purpose of clearly pointing out that the two approaches are not the same. More practically oriented, Section 4 calibrates the two different modeling approaches based on a so-called JDCEV driving process to the same observed market data and discusses the resulting differences. With a solid understanding of the differences between forward and spot modeling at hand, Section 5 discusses the basis for decision-making between the two approaches. In particular, since the possibility to include discrete cash dividends into the pricing model is one of the main advantages of the forward modeling approach, the respective technique of [3] to accomplish this task is reviewed and slightly enhanced as well in this section. Finally, Section 6 concludes, and a technical appendix explains two

important computations in the context of defaultable Markov diffusions for the readers' convenience.

2. Spot and forward model

As driving equity process for either share price (spot) or forward, we consider a defaultable Markov diffusion model,⁵ which is based on a stochastic process $\{Z_t^{(\mu)}\}$. The latter is given by

$$dZ_t^{(\mu)} = Z_t^{(\mu)} \left([\mu(t) + \lambda(Z_t^{(\mu)})] dt + \sigma(Z_t^{(\mu)}) dW_t \right),$$

with starting value denoted by $z_0 := Z_0^{(\mu)} > 0$, where $\mu : [0, \infty) \rightarrow \mathbb{R}$ is some deterministic drift function, and $\lambda, \sigma : \mathbb{R} \rightarrow (0, \infty)$ are smooth enough for the defining stochastic differential equation (SDE) to have a non-explosive weak solution under the constraint that the state zero, if attainable at all, is an absorbing boundary.⁶ For an exponential random variable ϵ with unit mean, that is independent of the Brownian motion $\{W_t\}$, we further define the random variable

$$\tau^{(\mu)} := \inf \left\{ t > 0 : \int_0^t \lambda(Z_s^{(\mu)}) ds > \epsilon \text{ or } Z_t^{(\mu)} = 0 \right\}.$$

In the sequel, we use the stochastic process $\{Z_t^{(\mu)} 1_{\{\tau^{(\mu)} > t\}}\}$ as a basis for modeling either the share price process ($\mu = r - \delta$) or the equity forward process ($\mu \equiv 0$), and then study the difference between both approaches. In particular, we are only interested in the diffusion process $Z_t^{(\mu)}$ until time $\tau^{(\mu)}$, when we pin it to zero. Since the model is used for pricing applications, it is defined directly under a risk-neutral probability measure. It is well-known that the stochastic process

$$e^{-\int_0^t \mu(s) ds} Z_t^{(\mu)} 1_{\{\tau^{(\mu)} > t\}}, \quad t \geq 0,$$

is a local martingale⁷ with respect to the filtration $\{\mathcal{F}_t\}$ that is generated jointly by the Brownian motion $\{W_t\}$ and the default indicator process $\{1_{\{\tau^{(\mu)} > t\}}\}$. This filtration is used as market filtration. The random variable $\tau^{(\mu)}$ is interpreted as the occurrence of a bankruptcy event, at which the share price and the equity forward drop to zero and remain there. The stochastic process $\{\lambda(Z_t^{(\mu)})\}$ is called *default intensity*, and intuitively

⁵Such models are treated, e.g., in [1],[2],[4]–[7],[10]–[12].

⁶Concrete examples satisfying these demands are given in Sections 3 and 4 below.

⁷For the reader's convenience, the Appendix provides a proof.

represents the instantaneous bankruptcy probability according to the first order approximation formula

$$\mathbb{P}(\tau^{(\mu)} \leq t + \Delta | \tau^{(\mu)} > t) \approx \lambda(Z_t^{(\mu)}) \Delta.$$

In light of this formula, the function $\lambda(\cdot)$ allows to model a reciprocal relationship between the value of $Z_t^{(\mu)}$ and the bankruptcy likelihood. Moreover, the function $\sigma(\cdot)$ allows to model a local volatility component. For the sake of generality, we do not further specify the functions $\lambda(\cdot)$ and $\sigma(\cdot)$ here, but provide parametric examples later. Depending on the choice of these model functions, the random variable $\tau^{(\mu)}$ may also be infinity, as Example 2.1 below shows.

2.1. Spot model

The drift function $\mu(\cdot)$ is interpreted as the difference between a risk-free discounting rate $r(\cdot)$ and a continuous yield $\delta(\cdot) \geq 0$ accounting for proceeds from stock possession, i.e. $\mu := r - \delta$. Based on the model $(\{Z_t^{(r-\delta)}\}, \tau^{(r-\delta)})$, the defaultable Markov diffusion model for the evolution of the share price $\{S_t\}_{t \geq 0}$ is the stochastic process defined as

$$S_t := Z_t^{(r-\delta)} \mathbf{1}_{\{\tau^{(r-\delta)} > t\}}, \quad t \in [0, T], \quad (6)$$

where T is a finite model horizon, and $z_0 := Z_0^{(r-\delta)} = S_0$. The model is in accordance with arbitrage pricing theory in the sense that the process

$$e^{-\int_0^t r(s) ds} e^{\int_0^t \delta(s) ds} S_t, \quad t \in [0, T],$$

which equals precisely the discounted wealth process from a portfolio that is long the stock, is a (non-negative) local martingale with respect to the market filtration $\{\mathcal{F}_t\}$. In particular, for $\delta \equiv 0$ the discounted share price process is a local martingale and for $\delta(\cdot) > 0$ it is a supermartingale. Within this share price model the equity forward process for $0 \leq t \leq T$ is given by

$$F(t, T) = \mathbb{E}[S_T | \mathcal{F}_t] = e^{\int_t^T (r(s) - \delta(s)) ds} Z_t^{(r-\delta)} \mathbf{1}_{\{\tau^{(r-\delta)} > t\}},$$

so that, in particular,

$$F(0, T) = e^{\int_0^T (r(s) - \delta(s)) ds} S_0. \quad (7)$$

Example 2.1 (Geometric Brownian motion). If $\lambda(z) \equiv \lambda \geq 0$ and $\sigma(z) \equiv \sigma > 0$ are both constant, then $\tau^{(r-\delta)} = \epsilon/\lambda$ has an exponential

distribution with mean $1/\lambda$, which is independent of $\{W_t\}$. The unique strong solution of the driving process SDE is given by

$$Z_t^{(\mu)} = S_0 e^{\int_0^t (r(s) - \delta(s)) ds + \left(\lambda - \frac{\sigma^2}{2}\right) t + \sigma W_t}, \quad t \geq 0,$$

so that $\log(Z_t^{(\mu)}/S_0)$ has a normal distribution with mean $\int_0^t (r(s) - \delta(s)) ds + \left(\lambda - \frac{\sigma^2}{2}\right) t$ and variance $\sigma^2 t$. The share price process is given by

$$S_t = S_0 e^{\int_0^t (r(s) - \delta(s)) ds + \left(\lambda - \frac{\sigma^2}{2}\right) t + \sigma W_t} 1_{\{\epsilon > \lambda t\}}.$$

Finally, we'd like to remark that in the special case $\lambda = 0$ bankruptcy cannot happen, i.e. $\tau^{(r-\delta)} = \infty$, and the model boils down to the famous Black-Scholes share price model, cf. [9].

2.2. Forward model

Instead of modeling the share price process exogenously and deriving the equity forward process, one may instead choose to model the equity forward process $\{F(t, T)\}_{t \in [0, T]}$ exogenously and derive the share price process $\{S_t\}_{t \in [0, T]}$ from it via the relation (1). The main motivation for this modeling approach is the separation of interest rate/ repo/ dividend modeling and stochastic modeling (captured in the model for $F(t, T)$), which are multiplicatively separated in (1). There is no need to consider $r(\cdot)$ and $\delta(\cdot)$ when modeling $F(t, T)$, because it is already incorporated adequately into the share price process according to Formula (1), one only needs to make sure that $\{F(t, T)\}_{t \in [0, T]}$ is a (local) martingale. The only thing which depends on $r(\cdot)$ and $\delta(\cdot)$ is the starting value $F(0, T)$ of the forward process. Consequently, basing the stochastic modeling of the equity forward process on the model $(\{Z_t^{(\mu)}\}, \tau^{(\mu)})$, it is natural to set $\mu \equiv 0$ and define

$$F(t, T) := Z_t^{(0)} 1_{\{\tau^{(0)} > t\}}, \quad t \in [0, T], \quad (8)$$

where $z_0 := Z_0^{(0)} = \exp\left(\int_0^T (r(s) - \delta(s)) ds\right) S_0$ in accordance with (7). This model for $F(t, T)$ depends on $r(\cdot)$ and $\delta(\cdot)$ only through its starting value z_0 . Summarizing, the resulting share price process in this approach is given by

$$S_t := e^{-\int_t^T (r(s) - \delta(s)) ds} Z_t^{(0)} 1_{\{\tau^{(0)} > t\}}, \quad t \in [0, T].$$

Example 2.2 (Geometric Brownian motion, contd.). Like in Example 2.1, assume that $\lambda(z) \equiv \lambda \geq 0$ and $\sigma(z) \equiv \sigma > 0$ are both constant.

Then we have

$$Z_t^{(0)} = S_0 e^{\int_0^T (r(s) - \delta(s)) \, ds} e^{(\lambda - \frac{\sigma^2}{2})t + \sigma W_t}, \quad 0 \leq t \leq T,$$

and $\tau^{(0)} = \epsilon/\lambda = \tau^{(r-\delta)}$ is an exponential random variable with mean $1/\lambda$, independent of $\{W_t\}$. Consequently,

$$\begin{aligned} S_t &= e^{-\int_t^T (r(s) - \delta(s)) \, ds} Z_t^{(0)} 1_{\{\tau^{(0)} > t\}} \\ &= S_0 e^{\int_0^t (r(s) - \delta(s)) \, ds + (\lambda - \frac{\sigma^2}{2})t + \sigma W_t} 1_{\{\epsilon > \lambda t\}}, \end{aligned}$$

which coincides exactly with the share price model in Example 2.1. Hence, for constant $\lambda(\cdot)$ and $\sigma(\cdot)$ the share price model approach and the forward model approach are equivalent. Finally, we'd like to remark that in the special case $\lambda = 0$ bankruptcy cannot happen, i.e. $\tau^{(0)} = \infty$, and the model for $F(t, T)$ boils down to the famous Black-model for the forward process, cf. [8]. In particular, the Black-forward model and the Black-Scholes share price model are equivalent.

2.2.1. Wealth process model

Modeling the forward process by (8) implies that the starting value of the driving process $z_0 = F(0, T)$ depends on the model horizon T . This is a drawback compared with the spot model approach (6), where the starting value of the driving process is simply the current share price $z_0 = S_0$, independent of T . In particular, when derivatives with different maturities have to be priced, the forward modeling approach has the disadvantage that different forward processes, with different time horizons and starting values, are required.

A third modeling approach, which shares with the forward model approach that the driving process is a (local) martingale, and which shares with the spot model approach the independence of the driving process' starting value of T , is to model the discounted wealth process $\{V_t\}_{t \in [0, T]}$ that arises from holding a portfolio which is long one share and continuously rehypothecates all proceeds from stock possession into more shares, given by

$$V_t := e^{-\int_0^t (r(s) - \delta(s)) \, ds} S_t = e^{-\int_0^T (r(s) - \delta(s)) \, ds} F(t, T).$$

Like in the forward model approach, since $\{V_t\}$ is a (local) martingale, a natural definition based on the model $(\{Z_t^{(\mu)}\}, \tau^{(\mu)})$ is to set $\mu \equiv 0$ and

$$V_t := Z_t^{(0)} 1_{\{\tau^{(0)} > t\}}, \quad t \in [0, T],$$

where $z_0 := Z_0^{(0)} = S_0$. Consequently, the share price in this setup is derived from the discounted wealth process via

$$S_t = e^{\int_0^t r(s) - \delta(s) ds} V_t, \quad t \in [0, T].$$

Remark 2.1 (Wealth modeling $\equiv$ forward modeling). *As the wealth process $\{V_t\}$ equals the forward process $\{F(t, T)\}$ up to a multiplicative constant which is independent of time t , the wealth process approach and the forward model approach are equivalent under the condition that the functions $\lambda(z)$ and $\sigma(z)$ depend on their argument z only through the fraction z/z_0 , i.e. default intensity and local volatility are independent of the starting value z_0 of the driving process. This is often satisfied in applications, and in particular in all examples within the present article. Consequently, we do not further distinguish between wealth process approach and forward model approach in the remainder of this article.*

3. First example: CEV model

As demonstrated in Example 2.2, the spot model approach is equivalent to the forward model approach if the functions $\lambda(\cdot)$ and $\sigma(\cdot)$ are constant. However, if they are not constant, both approaches imply different models, which is illustrated in Figure 1 below.

Figure 1 depicts two simulated paths of $\{S_t\}_{t \in [0, 20]}$, one simulated according to the spot model approach, the other according to the forward modeling approach. Both simulations are accomplished via an Euler scheme based on exactly the same path of the underlying Brownian motion. It is assumed that $\lambda \equiv 0$, but the local volatility component is modeled with a constant elasticity of variance (CEV) term $\sigma(z) = 0.6 \sqrt{z_0/z}$, cf. Delbaen and Shirakawa [15], who prove the existence of a non-explosive weak solution of the defining SDE, which also follows from the more general results in [10]. It is obvious from Figure 1 that the two modeling approaches are not equivalent. Most outstanding is the observation that the two paths have a different bankruptcy time. This difference can also be observed analytically as follows. For $\mu \geq 0$, the probability $\mathbb{P}(S_T = 0)$ is computed explicitly by Delbaen and Shirakawa [15]. In case of the spot modeling approach, it is given by

$$\mathbb{P}(S_T = 0) = \mathbb{P}(\tau^{(\mu)} \leq T) = \exp\left(-\frac{\mu}{0.18(1 - e^{-\mu T})}\right).$$

In particular, as the model horizon T tends to infinity, this probability tends to the value $\exp(-\mu/0.18)$, which is smaller than one for $\mu > 0$. This

Fig. 1. Share price process $\{S_t\}_{t \in [0,20]}$, once simulated from the spot model, and once from the forward model. The parameters are chosen as $\mu(t) = r(t) - \delta(t) \equiv 0.025$, $S_0 = 2$, with $\lambda(z) \equiv 0$, and $\sigma(z) = 0.6 \sqrt{z_0/z}$.

means that there is a positive probability that bankruptcy never occurs. In contrast, in case of the forward modeling approach, this probability is independent of μ and given by

$$\mathbb{P}(S_T = 0) = \mathbb{P}(F(T, T) = 0) = \mathbb{P}(\tau^{(0)} \leq T) = \exp\left(\frac{-1}{0.18T}\right),$$

which tends to one as the model horizon T tends to infinity. It is intuitive that the difference between spot and forward approach increases with increasing T , since the economic difference between spot and forward increases as well. Figure 2 depicts the probabilities $T \mapsto \mathbb{P}(S_T = 0)$ in dependence on T for the two different models.

4. Second example: JDCEV model

The CEV example of the previous section is rather academic in the following sense: comparing spot modeling approach and forward modeling approach with exactly the same parameters for their driving process is comparing

Fig. 2. The function $T \mapsto \mathbb{P}(S_T = 0)$ for the spot model and the forward model, with the same parameters as in Figure 1.

apples to oranges. In practice, either model is calibrated to one set of observed price data, so that the resulting spot model parameters and forward model parameters are typically different. In such a case, an investigation of the difference between both calibrated models is more delicate and interesting. In the sequel, we are going to carry out such a comparison for a particular model.

One of the most popular models for the joint pricing of credit- and equity-related derivatives is the so-called jump-to-default extended constant elasticity of variance (JDCEV) model of Carr and Linetsky [10], see also [1]. It extends the previously mentioned CEV model by a state-dependent default intensity term. More specifically, we consider the following parametric specification of the JDCEV model in the sequel:

$$\lambda(z) := \lambda_0 \left(\frac{z}{z_0} \right)^{2\beta}, \quad \sigma(z) := \sigma_0 \left(\frac{z}{z_0} \right)^{\beta},$$

with parameters $\lambda_0, \sigma_0 > 0$, and $\beta < 0$. The existence of a non-explosive weak solution for the defining SDE is shown in [10], who also explain that

the stopping time $\tau^{(\mu)}$ happens almost surely by the integrated default intensity exceeding ϵ , and not via diffusion of $\{Z_t^{(\mu)}\}$ through zero. Notice in particular that in our parameterization $\lambda(z)$ and $\sigma(z)$ depend on their argument z only via the fraction z/z_0 , so that the forward and wealth process modeling approaches are equivalent according to Remark 2.1. Furthermore, the power 2β in the default intensity is twice the power β of the local volatility, which is a deliberate choice making the model analytically tractable. In particular, with an arbitrary⁸ function $h : (0, \infty) \rightarrow \mathbb{R}$, it is shown in [10], Lemma 5.4, that

$$\mathbb{E}\left[h(Z_T^{(\mu)}) 1_{\{\tau^{(\mu)} > T\}}\right] = z_0 \mathbb{E}\left[\frac{h\left(e^{\int_0^T \mu(s) ds} (|\beta| \sqrt{cX})^{\frac{1}{|\beta|}}\right)}{(|\beta| \sqrt{cX})^{\frac{1}{|\beta|}}}\right], \quad (9)$$

where the auxiliary random variable X has a non-central χ^2 -distribution with non-centrality parameter $z_0^2 |\beta| / (c\beta^2)$ and $2\lambda_0 / (|\beta| \sigma_0^2) + 1/|\beta| + 2$ degrees of freedom, and

$$c = c(T, \sigma_0, z_0, \beta, \mu) = \frac{\sigma_0^2}{z_0^2 \beta} \int_0^T e^{-2|\beta| \int_0^u \mu(s) ds} du.$$

In particular, the default probability is computed from the last formula with $h \equiv 1$ as

$$\mathbb{P}(\tau^{(\mu)} \leq T) = 1 - z_0 \mathbb{E}\left[(|\beta| \sqrt{cX})^{-\frac{1}{|\beta|}}\right].$$

This default probability depends critically on the drift function $\mu(\cdot)$, hence on $r(\cdot) - \delta(\cdot)$, which directly carries over to the probability

$$\mathbb{P}(S_T = 0) = \mathbb{P}(\tau^{(\mu)} \leq T)$$

within the spot model approach. In contrast, for the forward model approach this default probability is independent of $\mu(\cdot)$ (and hence independent of both $r(\cdot)$ and $\delta(\cdot)$), since it is given by

$$\mathbb{P}(S_T = 0) = \mathbb{P}(\tau^{(0)} \leq T).$$

Consequently, interest rate and repo/ dividend parameters have a direct effect on default probabilities only within the spot price model.

We consider a credit default swap (CDS) par spread curve with maturities ranging from one year to ten years, referring to a company XY. We also consider European out-of-the-money (OTM) call and put options

⁸“Arbitrary” in the sense of a practitioner, of course, measurability issues swept under the carpet.

with maturity $T \approx 1.425$ years on the stock of company XY. The current stock price is $S_0 = 53.48$ and the forward $F(0, T) \approx 50.74$, implying a constant yield $\delta \approx 3.32\%$, where at the same time the discount rate $r(\cdot)$ has been bootstrapped in a piecewise constant manner from observed prices of interest rate swaps with a 3-month tenor. This example is inspired by a real-world case.

Fig. 3. Default probabilities for the next ten years in spot and forward model, both models being calibrated to the same CDS curve.

The JDCEV model parameters are fitted to the observed CDS quotes, which results in a perfect and unambiguous⁹ model fit in both cases. More precisely, the fitted parameters in spot (S) and forward (F) model¹⁰ are

⁹The observed CDS data already determines the model parameters uniquely.

¹⁰We actually use the wealth process modeling approach because we deal with different maturities, but according to Remark 2.1 the wealth process modeling approach is equivalent to the forward modeling approach.

given by

$$\begin{aligned}\beta_S &= -0.8, \sigma_S = 0.30, \lambda_S = 0.0077091, \\ \beta_F &= -0.6, \sigma_F = 0.39, \lambda_F = 0.0077091.\end{aligned}$$

Notice in particular that $\lambda_S = \lambda_F$, because this parameter can directly be read off from the short end of the observed CDS par spread curve. Both models calibrate perfectly to the observed CDS prices, which is illustrated by Figure 3 that depicts induced default probabilities for the next ten years. Notice that the CDS prices depend only on these probabilities, assuming a constant recovery rate, as usual. It is observed that both models imply almost perfectly the same default probabilities, hence CDS prices. Now what about the option data?

The distribution function of S_T has a jump at zero (corresponding to the default probability $\mathbb{P}(S_T = 0) > 0$) and is continuously differentiable on $(0, \infty)$. The top plot in Figure 4 shows the density function $f_S(\cdot; T) := \frac{\partial}{\partial x} \mathbb{P}(S_T \leq x)$ for $x > 0$, which is derived from Formula (9) in the Appendix. Whereas the expected value of S_T , which is the forward $F(0, T)$, is identical in both models, it is observed that the stock price has higher variance in the forward model. Consequently, the option prices in the forward model are higher than in the spot model, which is depicted in the bottom plot of Figure 4. The observed market prices of the options in the current example are perfectly in line with the option prices of the spot model, and are overestimated by the forward model.

Finally, Figure 5 visualizes the sensitivities of CDS prices on changes in the stock price with both considered JDCEV models. It is observed that the CDS spreads are more sensitive with regards to changes in the stock price within the stock price model, which is due to the fact that $|\beta_S| > |\beta_F|$.

Concluding, the two different models (spot and forward) differ significantly in the present use case. They differ with respect to the computation of price sensitivities, and they also differ in their projections of equity derivative prices from observed CDS prices, which is the most prominent application of such credit-equity models.

5. Implications for modeling

As demonstrated in the previous paragraph, when volatility and/ or default intensity in an equity model are allowed to depend on the current equity level, it makes a difference whether we decide to model the share price process (like in the Black–Scholes model), or instead we opt for modeling

Fig. 4. Top: The risk-neutral density function $f_S(\cdot; T)$ of S_T for $T \approx 1.425$ in spot and forward model. Bottom: Implied volatilities for European OTM calls and OTM puts with maturity $T \approx 1.425$ in spot and forward model.

Fig. 5. Effect of stock price change on the shape of a CDS par spread curve (computed with recovery rate 40% and running coupon 500 bps). Northwest: Stock price drops 80%. Northeast: Stock price drops 25%. Southwest: Stock price unchanged, current CDS par spread curve. Southeast: Stock price rises 25%.

the forward equity process (like in the Black–model). Moreover, Example 2.2 shows that this difference is not present in the classical Black–Scholes model with constant volatility (and zero default intensity), so it is an issue emerging only in advanced equity models, for instance with local volatility and/ or the possibility for a bankruptcy event sending the share price to zero. Since both spot and forward approach imply arbitrage-consistent pricing models, the decision between either one of them must be based on one’s opinion regarding the economic interpretation. Within the cosmos of defaultable Markov diffusion models considered in the present paper, the important question that needs to be answered is: “Do we want the dividend/ repo/ interest rate parameters to have influence on the default probabilities and local volatility of the equity process?” If the answer is “yes”, one prefers the spot model approach, if it is “no”, one prefers the forward model approach. If one does not care, it is also possible to work

with both models and prefer the one that fits observed price data better. For example, in the real-world case of the previous paragraph the market stock option data (which is not depicted in Figure 4) is explained better within the spot model. But the author wishes to mention that he could instead have easily presented another use case with the forward model being closer to the market data as well.

One of the most important advantages of the forward model over the spot model is the possibility to include discrete cash dividends in a simple manner, as described by Bernhart and Mai [3]. In this case, neglecting proceeds from stock lending¹¹ and focusing on proceeds from dividends only, a continuous dividend yield $\delta(\cdot)$ is not required, since dividends are modeled by discrete cash payments. If we want to include discrete cash dividends (which is straightforward in the forward model), but also want to make sure that interest rate and dividends have an influence on default probabilities (which requires the spot model), what can we do? We can use a hybrid model, using aspects of both discrete cash dividend modeling and continuous dividend modeling as follows. The discrete cash dividend model of [3] specifies a dividend payment at time t_k as a function $D_{t_k} = f(F(t_k, T))$ of the forward at t_k . The forward process $\{F(t, T)\}_{t \in [0, T]}$ is the driving process of the model, for instance being given by default intensity $\lambda(\cdot)$ and local volatility $\sigma(\cdot)$ according to the defaultable Markov diffusion considered in Section 4 of the present paper. The stock price is finally given by

$$S_t = F(t, T) e^{-\int_t^T r(s) ds} + \sum_{t < t_k \leq T} e^{-\int_t^{t_k} r(s) ds} \mathbb{E}[D_{t_k} | \mathcal{F}_t], \quad (10)$$

where the sum is taken over all dividend payment dates between t and T . Using Equation (1) with $t = 0$ and different forward maturities, the observable quantities S_0 and $\{F(0, \tilde{T})\}_{\tilde{T} \in [0, T]}$ may be used to extract a continuous dividend yield parameter $\delta(\cdot)$, for instance in a piecewise-constant manner with jumps at a discrete set of observed forward maturities $\tilde{T}$. This parameter $\delta(\cdot)$ may now be incorporated into the model (10) at one or both of the following two places:

¹¹Only for the sake of a simplified notation. It is easily possible to keep a part of δ as continuous yield from stock lending and only transform the remaining dividend part into discrete payments.

- (i) In order to make D_{t_k} dependent on both $r(\cdot)$ and $\delta(\cdot)$, we may change its definition to

$$D_{t_k} := f\left(e^{-\int_{t_k}^T r(s) - \delta(s) \, ds} F(t_k, T)\right).$$

- (ii) In order to make interest rate and dividends affect default probabilities and local volatility we change the default intensity and local volatility of the driving forward process from $\lambda(\cdot)$ and $\sigma(\cdot)$ to

$$\begin{aligned}\lambda_F(F(t, T), t) &:= \lambda\left(e^{-\int_t^T r(s) - \delta(s) \, ds} F(t, T)\right), \\ \sigma_F(F(t, T), t) &:= \sigma\left(e^{-\int_t^T r(s) - \delta(s) \, ds} F(t, T)\right).\end{aligned}$$

In particular, we introduce a dependence on the time t .

The inspiration for both adjustments (i) and (ii) above relies on the relation

$$S_t \approx e^{-\int_t^T r(s) - \delta(s) \, ds} F(t, T), \quad 0 \leq t \leq T, \quad (11)$$

which is no longer an equality in the presence of discrete cash dividends, but now only an approximation of the discrete relationship (10). The discrete payments cause jumps in the stock price process on the left-hand side of (11), whereas the right-hand side of (11) only represents a continuous proxy for the stock price (10), with the aforementioned definitions (i) and (ii) being in place. Altering the model of [3] in this way, it features jumps (of possibly random, state-dependent size) in the stock price process on dividend dates t_k , but there is also an effect of dividends and interest rates on default probability and local volatility, which is modeled continuously and non-random. In this sense, it is a hybrid model. The tractability of the model, which relies heavily on the fact that the forward modeling approach is chosen, is still huge, as can be seen from Equation (10): based on a stochastic differential equation for the driving forward process, standard finite difference techniques may be applied in a straightforward manner to compute the value of equity derivatives on the share price. The interested reader is referred to [3] for details.

6. Conclusion

We investigated the potential differences between basing an equity derivatives pricing model on either the share price or the equity forward. It has been demonstrated that it can be a crucial economic difference whether default intensity and local volatility of the driving process is modeled dependent on the spot-level or the forward-level. Since both approaches imply

arbitrage-consistent pricing models, the choice between them depends on the user's preference and economic interpretation.

Appendix A. Martingale property of driving process

We write $Z_t := \exp(-\int_0^t \mu(s) ds) Z_t^{(\mu)}$ and $\tau = \tau^{(\mu)}$, i.e. in particular we omit the superscript (μ) for notational simplicity. We furthermore introduce the notation $Y_t := Z_t 1_{\{\tau > t\}}$, $t \geq 0$, and aim to prove that $\{Y_t\}$ is a local martingale with respect to the market filtration that is generated by the Brownian motion and the observable default indicator. Fix $T \geq t \geq 0$. We compute the dynamics of the process $\left\{ Z_u \exp\left(-\int_t^u \lambda(Z_s) ds\right) \right\}_{u \geq t}$ and observe

$$\begin{aligned} d\left(Z_u e^{-\int_t^u \lambda(Z_s) ds}\right) &= e^{-\int_t^u \lambda(Z_s) ds} dZ_u - \lambda(Z_u) Z_u e^{-\int_t^u \lambda(Z_s) ds} du \\ &= e^{-\int_t^u \lambda(Z_s) ds} Z_u \sigma(Z_u) dW_u. \end{aligned}$$

Consequently, the latter process is driftless, showing that itself is a local martingale with respect to the natural filtration $\{\mathcal{F}_t^{(W)}\}_{t \geq 0}$ of the Brownian motion W . Hence, there exists an increasing sequence of $\{\mathcal{F}_t^{(W)}\}$ -stopping times $(T_n)_{n \in \mathbb{N}}$, such that $\lim_{n \rightarrow \infty} T_n = \infty$ and the stopped processes are martingales. Since $\mathcal{F}_t^{(W)} \subset \mathcal{F}_t$, $(T_n)_{n \in \mathbb{N}}$ is also a sequence of $\{\mathcal{F}_t\}$ -stopping times. We denote by $Y^{[T_n]} := \{Y_{T_n \wedge t}\}_{t \geq 0}$ the process Y stopped at T_n , and show that $Y^{[T_n]}$ is a martingale for each n .

$$\begin{aligned} \mathbb{E}[Y_T^{[T_n]} | \mathcal{F}_{t \wedge T_n}] &= \mathbb{E}[Z_{T \wedge T_n} 1_{\{\tau > T \wedge T_n\}} | \mathcal{F}_{t \wedge T_n}] \\ &= 1_{\{\tau > t \wedge T_n\}} \mathbb{E}[Z_{T \wedge T_n} 1_{\{\tau > T \wedge T_n\}} | \mathcal{F}_{t \wedge T_n}] \\ &= 1_{\{\tau > t \wedge T_n\}} \mathbb{E}[Z_{T \wedge T_n} \mathbb{E}[1_{\{\tau > T \wedge T_n\}} | \mathcal{F}_{t \wedge T_n} \vee \mathcal{F}_T^{(W)}] | \mathcal{F}_{t \wedge T_n}] \\ &= 1_{\{\tau > t \wedge T_n\}} \mathbb{E}[Z_{T \wedge T_n} \mathbb{E}[1_{\{\int_0^{T \wedge T_n} \lambda(Z_s) ds < \epsilon\}} | \mathcal{F}_{t \wedge T_n} \vee \mathcal{F}_T^{(W)}] | \mathcal{F}_{t \wedge T_n}] \\ &\stackrel{(*)}{=} 1_{\{\tau > t \wedge T_n\}} \mathbb{E}[Z_{T \wedge T_n} \mathbb{E}[e^{-\int_{t \wedge T_n}^{T \wedge T_n} \lambda(Z_s) ds} | \mathcal{F}_{t \wedge T_n} \vee \mathcal{F}_T^{(W)}] | \mathcal{F}_{t \wedge T_n}] \\ &= 1_{\{\tau > t \wedge T_n\}} \mathbb{E}[Z_{T \wedge T_n} e^{-\int_{t \wedge T_n}^{T \wedge T_n} \lambda(Z_s) ds} | \mathcal{F}_{t \wedge T_n}] \stackrel{(**)}{=} 1_{\{\tau > t \wedge T_n\}} Z_{t \wedge T_n} = Y_t^{[T_n]}, \end{aligned}$$

where Equality $(*)$ follows from the lack-of-memory property of the exponential distribution

$$\mathbb{P}(\epsilon > b | \epsilon > a) = \mathbb{P}(\epsilon > b - a), \quad b \geq a \geq 0,$$

and $(**)$ follows from the fact that $\left\{ Z_u \exp\left(-\int_t^u \lambda(Z_s) ds\right) \right\}_{u \geq t}$ is a local martingale.

Appendix B. Density of S_T in JDCEV model

We briefly demonstrate how to derive the probability law of S_T from Formula (9). First of all, we consider the following bijection on $(0, \infty)$:

$$g(x) = (|\beta| \sqrt{cx})^{\frac{1}{|\beta|}}, \quad g^{-1}(s) = s^{2|\beta|} \frac{1}{c\beta^2}, \quad g'(x) = \frac{1}{2} \sqrt{\frac{c}{x}} (|\beta| \sqrt{cx})^{\frac{1}{|\beta|}-1}.$$

Formula (9) states that

$$\mathbb{E} \left[h(Z_T^{(\mu)}) 1_{\{\tau^{(\mu)} > T\}} \right] = z_0 \mathbb{E} \left[\frac{h(e^{\int_0^T \mu(s) ds} g(X))}{g(X)} \right],$$

where the random variable X has density f_X of a certain non-central χ^2 -distribution. Consequently, we observe with the change of variables $u = e^{\int_0^T \mu(s) ds} g(x)$ that

$$\begin{aligned} \mathbb{E} \left[h(Z_T^{(\mu)}) 1_{\{\tau^{(\mu)} > T\}} \right] &= z_0 \int_0^\infty \frac{h(e^{\int_0^T \mu(s) ds} g(x))}{g(x)} f_X(x) dx \\ &= z_0 \int_0^\infty \frac{h(u) f_X(g^{-1}(u e^{-\int_0^T \mu(s) ds}))}{u g'(g^{-1}(u e^{-\int_0^T \mu(s) ds}))} du \\ &= \int_0^\infty h(u) \underbrace{\frac{2 z_0 f_X\left(\frac{(u e^{-\int_0^T \mu(s) ds})^{2|\beta|}}{c \beta^2}\right)}{u c |\beta| \left(u e^{-\int_0^T \mu(s) ds}\right)^{1-2|\beta|}}}_{=: f_Z(u; T, \mu)} du. \end{aligned}$$

In the spot model we have $S_t = Z_t^{(\mu)} 1_{\{\tau^{(\mu)} > 0\}}$, hence

$$\begin{aligned} \mathbb{E} \left[h(S_T) \right] &= \mathbb{E} \left[h(S_T) 1_{\{S_T > 0\}} \right] + h(0) \mathbb{P}(S_T = 0) \\ &= \mathbb{E} \left[h(Z_T^{(\mu)}) 1_{\{\tau^{(\mu)} > T\}} \right] + h(0) \mathbb{P}(\tau^{(\mu)} \leq T) \\ &= \int_0^\infty h(u) f_Z(u; T, \mu) du + h(0) \left(1 - \int_0^\infty f_Z(u; T, \mu) du \right), \end{aligned}$$

which shows that $f_Z(\cdot; T, \mu) / \int_0^\infty f_Z(u; T, \mu) du$ equals the density of S_T conditioned on the event $\{S_T > 0\}$. The complementary event $\{S_T = 0\}$ has probability $1 - \int_0^\infty f_Z(u; T, \mu) du$. In other words, $f_S(x; T) := \frac{\partial}{\partial x} \mathbb{P}(S_T \leq x) = f_Z(x; T, \mu)$ for $x > 0$.

In the wealth process model, which is equivalent to the forward model by Remark 2.1, we have $S_t = e^{\int_0^t r(s) - \delta(s) ds} Z_t^{(0)} 1_{\{\tau^{(0)} > t\}}$, hence the same

computation as above shows

$$\begin{aligned}\mathbb{E}\left[h(S_T)\right] &= \mathbb{E}\left[h(S_T) 1_{\{S_T > 0\}}\right] + h(0) \mathbb{P}(S_T = 0) \\ &= \mathbb{E}\left[h\left(e^{\int_0^T r(s) - \delta(s) ds} Z_T^{(0)}\right) 1_{\{\tau^{(0)} > T\}}\right] + h(0) \mathbb{P}(\tau^{(0)} \leq T) \\ &= \int_0^\infty h\left(e^{\int_0^T r(s) - \delta(s) ds} u\right) f_Z(u; T, 0) du + h(0) \left(1 - \int_0^\infty f_Z(u; T, 0) du\right).\end{aligned}$$

Substituting $x = e^{\int_0^T r(s) - \delta(s) ds} u$, this implies for $x > 0$ that

$$f_S(x; T) = \frac{\partial}{\partial x} \mathbb{P}(S_T \leq x) = e^{-\int_0^T r(s) - \delta(s) ds} f_Z(e^{-\int_0^T r(s) - \delta(s) ds} x; T, 0).$$

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