Innovations in Insurance, Risk- and Asset Management

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Chapter 11

Liability Driven Investments with a Link to Behavioral Finance

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Liability driven investment (LDI) strategies that take stochastic liabilities into account have become increasingly important for insurance companies and pension funds due to market developments such as low interest rates, high volatility and changes in regulatory requirements. We consider stochastic liabilities in a portfolio optimization framework and include aspects from behavioral finance, in particular cumulative prospect theory (CPT). We study LDI strategies with extended preference structures and probability distortion and derive analytical solutions for a CPT portfolio optimization problem in an LDI context. Within a case study, we compare the optimal investment strategies to existing LDI approaches within traditional frameworks such as the partial surplus optimization presented in [1] and the funding ratio optimization in an expected utility framework as introduced in [2].

Keywords: asset liability management, liability driven investments, cumulative prospect theory.

1. Introduction

In the literature on intertemporal portfolio choice, the optimization of the utility of terminal wealth and consumption as firstly presented in [3] is a widely accepted standard. However, according to regulatory standards, insurance companies and pension funds, e.g., have to consider the level of wealth relative to the value of the liabilities. Consequently, liabilities were included in several different ways into portfolio optimization frameworks in discrete and continuous time. In [1], the authors provide an extension of the mean-variance approach for a one-period setting. The authors of [4] extend the mean-variance framework by considering the downside risk inherent in the liabilities in form of an exchange option and [5] provide a continuous-time version of the surplus management approach from [1]. In a continuous-time model, [6], [7] and [8] consider generalizations of CPPI strategies with a stochastic floor that can be interpreted as stochastic liabilities. In [2], the

author argues that what matters is the relative wealth of the asset portfolio with respect to the value of the liabilities and optimizes the terminal wealth of the funding ratio. He develops optimal strategies for a model in which both, the assets and the liabilities, are modeled as geometric Brownian motions. With the recent international accounting standards IFRS (especially IAS 19) for pension funds and the European regulatory requirements Solvency II for insurance companies, the importance of investment strategies that are adapted to stochastic liabilities, has increased. However, this short overview illustrates that no widely accepted scientific standard exists how liabilities should be included in the portfolio optimization.

For the market model from [2], we generalize previous results by embedding funding ratio optimization in a cumulative prospect theory (CPT) framework. The CPT framework provides two further enrichments. Firstly, a probability distortion is included. Within the CPT framework, distortions are used to create subjective probabilities of events. However, distortions can also be used to model heavy-tailed portfolio returns. Secondly, the utility function is extended and assumes risk-seeking behavior in case of underfunding. This extension also contributes to the literature that especially considers underfunded plans, e.g. [9] deal with underfunded plans in a discrete setting.

The article is structured as follows: In Section 2, we introduce the model for assets and liabilities. Optimal investment strategies for the funding ratio are described in Section 3 for an expected utility framework. In Section 4, we consider the optimization of the funding ratio within cumulative prospect theory. This approach is compared to the approach from Section 3 and an application of the approach described in [1] in Section 5. A conclusion is provided in Section 6.

2. A model for assets and liabilities

We introduce a market model on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ with $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$ being the filtration generated by the n -dimensional standard Brownian motion $\tilde{W} := (W^1, \dots, W^n)'$, satisfying the usual conditions. Following [2], we use $W := (W^1, \dots, W^{n-1})'$ to model the assets and $W^\epsilon := W^n$ to model the part of the liability risk that cannot be hedged. That means we have $n - 1$ risky assets with price processes $P = (P^1, \dots, P^{n-1})'$ and one risk-free asset with price process P^0 . The assets are assumed to follow a geometric Brownian motion with

$$dP_t^i = P_t^i (\mu_i dt + \sigma_i dW_t), \quad i \in \{1, 2, \dots, n - 1\},$$

$P_0^i = 1$, $i \in \{1, 2, \dots, n-1\}$ and $dP_t^0 = P_t^0 r dt$, $P_0^0 = 1$. The volatility matrix σ with rows σ_i , $i \in \{1, \dots, n-1\}$ is assumed to be positive definite. With $\mathbf{1} = (1, 1, \dots, 1)' \in \mathbb{R}^{n-1}$, the market price of risk is then given by

$$\theta := \sigma^{-1}(\mu - r\mathbf{1}).$$

We consider $\mathcal{F}_t$ -progressively-measurable, self-financing investment strategies $(\varphi_t^0, \varphi_t)_{t \in [0, T]}$ with $\varphi_t = (\varphi_t^1, \dots, \varphi_t^{n-1})'$, φ_t^i being the fraction of wealth invested in asset i at time t satisfying

$$\int_0^T (A_t^\varphi)^2 \|\varphi_t\|^2 dt < \infty \quad \mathbb{P} - a.s.,$$

and A_t^φ denoting the wealth process associated with $(\varphi_t^0, \varphi_t)_{t \in [0, T]}$. Its dynamics are given by

$$dA_t^\varphi = A_t^\varphi [(r + \varphi_t'(\mu - r\mathbf{1}))dt + \varphi_t' \sigma dW_t].$$

The initial wealth A_0 is assumed to be non-negative. By $\Lambda(t, v)$, we denote the set of all admissible investment strategies on $[t, T]$ with $A_t^\varphi = v$ and such that $\mathbb{E}[U(A_T^\varphi) | \mathcal{F}_t]$ is well-defined. Furthermore, let $\Lambda := \Lambda(0, v)$. The performance seeking portfolio (growth optimal portfolio) is known from portfolio optimization without liabilities and logarithmic utility and is defined as

$$\varphi^{PS} := (\sigma \sigma')^{-1}(\mu - r\mathbf{1}). \quad (1)$$

As in [10] and [2], we model the liability process L^ϵ as a geometric Brownian motion following the dynamics

$$dL_t^\epsilon = L_t^\epsilon (\mu_L dt + \sigma_L dW_t + \sigma_\epsilon dW_t^\epsilon), \quad L_0 = 1,$$

with drift μ_L , hedgeable risks related to W and non-hedgeable risks related to W^ϵ , independent of W . In the context of an insurance company or pension fund, the non-hedgeable risks could be, e.g., actuarial risks like longevity risk or underwriting risk. If we have $\sigma_\epsilon = 0$, all the liability risks emerge from W and the liabilities can be hedged. The liability hedging portfolio as introduced in [2] is then given by

$$\varphi^{LH} := (\sigma')^{-1} \sigma_L'. \quad (2)$$

In Section 3, we assume $\sigma_\epsilon \geq 0$ to show that in the expected utility framework, the optimal strategy is independent of the non-hedgeable risks. This observation provides the reasoning for the assumption $\sigma_\epsilon = 0$ which we

need in Section 4. Instead of considering the absolute wealth A_t^φ , we consider the funding ratio defined as the relative wealth with respect to the value of the liabilities

$$X_t^\varphi := \frac{A_t^\varphi}{L_t^\epsilon}, \quad t \in [0, T]$$

for a corresponding portfolio process φ . By an application of Ito's formula, we get the mean return and volatility of the funding ratio

$$\begin{aligned} \mu_X^\varphi(t) &= r + \varphi'_t(\mu - r\mathbf{1}) - \mu_L + \sigma_L \sigma'_L + \sigma_\epsilon^2 - \sigma_L \sigma'_\varphi t, \\ \sigma_X^\varphi(t) &= \left((\varphi'_t \sigma - \sigma_L) (\varphi'_t \sigma - \sigma_L)' + \sigma_\epsilon^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We define a one-dimensional Brownian Motion $\bar{W}_t$, such that

$$\sigma_X^\varphi(t) \bar{W}_t = (\varphi'_t \sigma - \sigma_L) W_t - \sigma_\epsilon W_t^\epsilon.$$

The funding ratio thus follows the SDE

$$dX_t^\varphi = X_t^\varphi [\mu_X^\varphi(t) dt + \sigma_X^\varphi(t) d\bar{W}_t], \quad t \in [0, T]$$

or equivalently

$$dX_t^\varphi = X_t^\varphi [\mu_X^\varphi(t) dt + (\varphi'_t \sigma - \sigma_L) dW_t - \sigma_\epsilon dW_t^\epsilon] \quad t \in [0, T].$$

It is visible in this representation that the liability hedging portfolio φ^{LH} is the portfolio that hedges the liabilities best in the sense that the volatility of the funding ratio is minimized. Let $\sigma_\epsilon = 0$ for the rest of the section and $L = L^0$ denote the corresponding liability process. Then, there exists a unique risk-neutral measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, defined by the Radon-Nikodym derivative (see e.g. [11], Chapter 5)

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\theta'\theta t - \theta'W_t\right)$$

and $\frac{A_t^\varphi}{P_t^0}$ is a $\mathbb{Q}$ -martingale. The Pricing Kernel is then given by

$$\tilde{Z}_t := (P_t^0)^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-rt - \theta'W_t - \frac{1}{2}\theta'\theta t\right).$$

We change the numéraire from the risk-free asset to the liability process L . To do this, let $\mathbb{Q}^L$ be the risk-neutral measure under the numéraire L . Then, $X_t^\varphi = \frac{A_t^\varphi}{L_t}$ is a $\mathbb{Q}^L$ -martingale and the new pricing kernel is given by

$$Z_t^L := \frac{d\mathbb{Q}}{d\mathbb{P}} \frac{d\mathbb{Q}^L}{d\mathbb{Q}}|_{\mathcal{F}_t} = \frac{L_t}{L_0 P_t^0} \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\|\sigma_L - \theta'\|^2 t + (\sigma_L - \theta')W_t\right). \quad (3)$$

Since we only work with this pricing kernel, we set $Z_t := Z_t^L$ and $Z := Z_T^L$. The distribution function of Z is given by

$$F_Z(z) = \mathcal{N}\left(\frac{\ln z + \frac{1}{2}\|\sigma'_L - \theta\|^2 T}{\|\sigma'_L - \theta\|\sqrt{T}}\right) \quad (4)$$

and its quantile function by

$$F_Z^{\leftarrow}(p) = \exp\left(-\frac{1}{2}\|\sigma_L - \theta'\|^2 T + \|\sigma_L - \theta'\|\sqrt{T}\mathcal{N}^{\leftarrow}(p)\right),$$

with $\mathcal{N}$ and $\mathcal{N}^{\leftarrow}$ denoting the distribution function and the quantile of the standard normal distribution.

3. Expected utility framework

3.1. The optimization problem

In this section, we go back to the case of possible non-hedgeable risks, so we assume $\sigma_\epsilon \geq 0$ and find an investment strategy φ that solves the optimization problem

$$\sup_{\varphi \in \Lambda} \mathbb{E}[U(X_T^\varphi)]$$

for a strictly increasing, strictly concave and twice continuously differentiable utility function satisfying

$$\lim_{x \downarrow 0} U'(x) = \infty, \quad \lim_{x \rightarrow \infty} U'(x) = 0,$$

where we will mainly use the power utility function of the form

$$U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma \in (-\infty, 0) \cup (0, 1).$$

The value function is defined as

$$\Phi(t, v) := \sup_{\varphi \in \Lambda(t, v)} \mathbb{E}[U(X_T^\varphi) | F_t]. \quad (5)$$

The following result, which can be found in [2], states an optimal investment strategy for the introduced funding ratio optimization problem.

Theorem 3.1 (Three-Fund Separation, Expected Utility Theory).

The optimal investment strategy φ^* is given by

$$\varphi^*(t, X_t^\varphi) = (1 - \lambda^{EU}(t, X_t^\varphi)) \varphi^{LH} + \lambda^{EU}(t, X_t^\varphi) \varphi^{PS}$$

with $\lambda^{EU}(t, X_t^\varphi)$ being the inverse of the Arrow-Pratt measure of relative risk aversion (relative risk tolerance) of $\Phi(t, X_t^\varphi)$

$$\lambda^{EU}(t, X_t^\varphi) := - \frac{\frac{\partial}{\partial X_t^\varphi} \Phi(t, X_t^\varphi)}{X_t^\varphi \frac{\partial^2}{\partial^2 X_t^\varphi} \Phi(t, X_t^\varphi)},$$

the liability hedging portfolio as in (2) and the performance seeking portfolio from (1). The remaining fraction of wealth $1 - \varphi^*(t, X_t^\varphi)' \mathbf{1}$ is invested in the risk-free asset.

Proof. The Hamilton-Jacobi-Bellman (HJB) equation associated with (5) can be written as:

$$\frac{\partial}{\partial t} \Phi(t, v) + \sup_{\varphi \in \Lambda(t, v)} \left[\mu_X^\varphi(t) v \frac{\partial}{\partial v} \Phi(t, v) + \frac{1}{2} (\sigma_X^\varphi(t))^2 v^2 \frac{\partial^2}{\partial^2 v} \Phi(t, v) \right] = 0,$$

with terminal condition $\Phi(T, v) = U(v)$, $v \in (0, \infty)$. If there exists a solution to the HJB equation, the optimal strategy can be obtained by calculating the supremum in the HJB equation. For general $\Phi(t, v)$, we obtain the optimal strategy by maximizing the help function

$$M(c) := \mu_X^c(t) \cdot v \cdot \frac{\partial}{\partial v} \Phi(t, v) + \frac{1}{2} \cdot (\sigma_X^c(t))^2 \cdot v^2 \cdot \frac{\partial^2}{\partial^2 v} \Phi(t, v).$$

The first order condition reads

$$((\mu - r\mathbf{1}) - \sigma\sigma'_L) v \frac{\partial}{\partial v} \Phi(t, v) + (\sigma\sigma'c - \sigma\sigma'_L) v^2 \frac{\partial^2}{\partial^2 v} \Phi(t, v) = 0$$

and holds for

$$c^* = (1 - \lambda^{EU}(t, v))(\sigma')^{-1} \sigma_L + \lambda^{EU}(t, v)(\sigma')^{-1} \theta$$

and thus, the optimal investment strategy is given by

$$\varphi^*(t, X_t^\varphi) = (1 - \lambda^{EU}(t, X_t^\varphi)) \varphi^{LH} + \lambda^{EU}(t, X_t^\varphi) \varphi^{PS}.$$

□

As in [2], we get a three-fund separation theorem, with the funds being the performance seeking portfolio, liability hedging portfolio and risk-free asset.

Corollary 3.1 (Three-Fund Separation, Power Utility). *For the power utility, we get*

$$\lambda^{EU}(t, X_t^\varphi) = \frac{1}{1 - \gamma}.$$

Proof. The statement directly follows with Theorem 3.1 and Appendix A. $\square$

We see that the optimal strategy does not depend on the non-hedgeable risks associated with σ_ϵ for the power utility. To cover these risks, an additional capital buffer has to be used. Therefore, we assume in the following that a certain part of the wealth is used for this purpose and we only deal with the hedgeable risks here, i.e. $\sigma_\epsilon = 0$ in the following section.

4. Extension to cumulative prospect theory

4.1. The optimization problem

The funding ratio optimization can be extended further to Cumulative Prospect Theory (CPT) using the quantile optimization approach by [12]. As it requires a complete market, we set $\sigma_\epsilon = 0$ in this section.

Cumulative Prospect Theory is an extension of traditional Expected Utility Theory introduced in [13] and [14]. Its key innovations are a separate treatment of gains and losses with respect to utility and the introduction of a probability distortion function.

Experiments presented in [13] show that people are risk averse with respect to gains and risk friendly when confronted with losses where gains and losses are separated by a reference point. Hence, in case of losses, a convex utility function is used which implies a risk-seeking behavior of the investor. For a pension plan, this could be the case when it is underfunded and aims to eliminate the deficit. In this currently serious situation, the company might be willing to take more risk than in a situation in which the plan is overfunded and the company does not want to put the good funding status at risk.

The probability distortion has two possible interpretations. The first interpretation is the one normally seen for probability distortion functions in a CPT framework. Here, the probability distortion functions are used to model the observation that people overestimate small probabilities and underestimate bigger probabilities. Thus probability distortion functions model an investor-specific, irrational bias. Alternatively, the probability distortion function can also be seen as a property of the asset portfolio

rather than the investor. By distorting the probabilities, heavy tails in the distribution of asset returns that are not modeled in a Black-Scholes market can be included and the distortion parameters can thus be fitted using market data. Technical conditions preclude the use of the classical probability distortion functions in this context from [14] and [15]. The authors of [12] introduce their own probability distortion function, however, this distortion function has a large amount of parameters. To simplify the fitting of the parameters and their interpretation, we present an extension of the distortion function introduced in [16] that still allows us to use the quantile optimization approach introduced in [12]. It is reverse-S shaped and has only two parameters. We begin by introducing the optimization approach following [12].

Wealth is considered relative to a *reference point* B , which we assume to be constant in our application. A *CPT utility function* $U : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$U(x) := U_+((x - B)^+) \mathbf{1}_{x \geq B}(x) - U_-((x - B)^-) \mathbf{1}_{x < B}(x)$$

where

$$(x)^+ = \max\{x, 0\}, (x)^- = -\min\{x, 0\}.$$

and $U_+, U_- : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ are utility functions as in the previously presented expected utility framework with $U_+(0) = U_-(0) = 0$ and $\forall x \in \mathbb{R}^+ : U'_-(x) > U'_+(x)$. The *probability distortion functions* w_+, w_- are required to be twice differentiable, strictly increasing and satisfy:

$$w_{\pm} : [0, 1] \rightarrow [0, 1], \quad w_{\pm}(0) = 0, \quad w_{\pm}(1) = 1, \quad w'_{\pm} > 0.$$

We set $X := X_T^{\varphi}$. The reference point will be set in such a way that it marks the point where the pension plan is considered to be adequately funded. A natural reference point when optimizing the funding ratio is where the plan is fully funded, i.e. for $B = 1$. We optimize the funding ratio by performing a change of numéraire from the risk-free asset P_T^0 to L_T . Thus we will use the pricing kernel from (3).

To be able to apply the approach from [12], we need to assume that the following *monotonicity condition* holds:

$$\frac{F_Z^{\leftarrow}(y)}{w'_+(y)} \text{ is non-decreasing for } y \in (0, 1]. \quad (\mathbf{M})$$

In Section 4.2, we introduce and work with a distortion that fulfills **(M)**. This is shown explicitly in Lemma 4.1. Using the defined CPT-utility functions and the probability distortion functions, the two value functions for gains and losses are given by

$$\begin{aligned}\mathbb{V}_+(\bar{X}) &:= \mathbb{E} [U_+(\bar{X})w'_+(1 - F_{\bar{X}}(\bar{X}))] = \int_0^\infty w_+(\mathbb{P}(U_+(\bar{X}) > x))dx \\ \text{and } \mathbb{V}_-(\bar{X}) &:= \mathbb{E} [U_-(\bar{X})w'_-(1 - F_{\bar{X}}(\bar{X}))] = \int_0^\infty w_-(\mathbb{P}(U_-(\bar{X}) > y))dy,\end{aligned}$$

where $\bar{X} := X - B$ and $F_{\bar{X}}$ denotes the distribution function of $\bar{X}$. Fitting them together, we define the value function $\mathbb{V}$ as

$$\mathbb{V}(\bar{X}) := \mathbb{V}_+((\bar{X})^+) - \mathbb{V}_-((\bar{X})^-).$$

The optimization problem is then given by:

$$\begin{aligned}\sup_{\bar{X}} \mathbb{V}(\bar{X}) \\ \text{s.t. } \mathbb{E}[Z\bar{X}] &= \frac{A_0}{L_0} - B\end{aligned}\tag{OP}$$

$\bar{X}$ is $\mathcal{F}_T$ -measurable and lower-bounded.

To find an optimal solution for **(OP)**, we separate **(OP)** into a Gains Problem and a Loss Problem depending on parameters c and v_+ and then find an optimal combination of those parameters in the so-called Gluing Problem. The optimal solution found by this approach is then equivalent to the optimal solution of the Original Problem, as stated in Theorem 4.1. The details of this optimization approach can be found in Appendix B.

Using this setup, the authors of [12] show a way to solve the optimization problem explicitly in the case of power utility functions. We adapt their results to an optimization of the funding ratio. In this section, we work with power utility functions defined as

$$U_+(x) = x^\gamma, \quad U_-(x) = \beta x^\gamma, \quad \gamma \in (0, 1), \quad \beta > 1.$$

We define

$$\begin{aligned}G(c) &:= \mathbb{E} \left[Z^{\frac{\gamma}{\gamma-1}} w'_+(F_Z(Z))^{\frac{1}{1-\gamma}} \mathbf{1}_{Z \leq c} \right], \\ \text{and } k(c) &:= \frac{\beta w_-(1 - F_Z(c))}{G(c)^{1-\gamma} \mathbb{E}[Z \mathbf{1}_{Z > c}]^\gamma}.\end{aligned}$$

The following result, adapted from [12], provides explicit solutions depending on the initial funding $\frac{A_0}{L_0}$ of the investor. We assume that the following condition holds:

$$\inf_{c>0} k(c) \geq 1. \quad (\mathbf{K})$$

Theorem 4.1 (Optimal Payoffs, CPT). *Let (M) and (K) be satisfied. If we start at least funded, i.e. $\frac{A_0}{L_0} \geq B$, then the following holds:*

- The optimal solution $\bar{X}^*$ to (OP) is given by

$$\bar{X}^* = (\bar{X}^*)^+ = \frac{\frac{A_0}{L_0} - B}{G(\infty)} \left(\frac{Z}{w'_+(F_Z(Z))} \right)^{\frac{1}{\gamma-1}}$$

$$\text{and } X^* = \bar{X}^* + B.$$

If we start underfunded, i.e. $\frac{A_0}{L_0} < B$, then the following holds:

- If $\inf_{c>0} k(c) = 1$, the supremum value of (OP) is 0 but not attainable.
- If $\inf_{c>0} k(c) > 1$, the Problem (OP) admits an optimal solution if and only if the problem

$$\inf_{0 \leq c < \infty} \left(\frac{\beta w_-(1 - F_Z(c))}{\mathbb{E}[Z \mathbf{1}_{Z>c}]^\gamma} \right)^{\frac{1}{1-\gamma}} - G(c) \quad (\mathbf{C})$$

admits an optimal solution c^* .

- If $c^* = 0$ is the only solution to (C), then

$$\bar{X}^* = \frac{A_0}{L_0} - B$$

$$\text{and } X^* = \bar{X}^* + B = \frac{A_0}{L_0}.$$

- If $c^* > 0$, the optimal solution to (OP) is given by

$$\begin{aligned} \bar{X}^* &= (\bar{X}^*)^+ - (\bar{X}^*)^- \\ &= \frac{v_+^*}{G(c^*)} \left(\frac{Z}{w'_+(F_Z(Z))} \right)^{\frac{1}{\gamma-1}} \mathbf{1}_{Z \leq c^*} - \frac{v_+^* - \frac{A_0}{L_0} + B}{\mathbb{E}[Z \mathbf{1}_{Z>c^*}]} \mathbf{1}_{Z>c^*} \end{aligned}$$

with

$$v_+^* = \frac{B - \frac{A_0}{L_0}}{k(c^*)^{1/(1-\gamma)} - 1}$$

$$\text{and } X^* = \bar{X}^* + B.$$

Proof. The proof works along [12] with the wealth being replaced by the funding ratio and the change of numéraire as described in Section 2. $\square$

To further specify the optimal solutions and the well-posedness of the optimization problem, probability distortion functions are introduced in the following sections. Using the completeness of the market, we will also derive replicating strategies for the optimal pay-offs.

4.2. Probability distortion function

While¹

$$\bar{w}(p) := \int_0^{F_Z^{\leftarrow}(p)} r f_Z(r) dr = \mathbb{E} \left[Z \mathbf{1}_{Z \leq F_Z^{\leftarrow}(p)} \right] = \mathcal{N} \left(\mathcal{N}^{\leftarrow}(p) - \|\sigma_L - \theta'\| \sqrt{T} \right)$$

with f_Z being the density function of Z , is by definition a probability distortion function, it is not reverse-S-shaped but rather convex. The proof can be found in Appendix C in Lemma C.1.

In order to get reverse S-shaped distortion functions, we introduce an alteration and define the probability distortion function we will work with as

$$w(p) := \mathcal{N}^\alpha \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right),$$

where $\alpha \in (0, 1]$ and $\delta \in (0, 1]$ are parameters that can be chosen to adjust the distortion behavior of w and $\mathcal{N}^\alpha(\cdot) := (\mathcal{N}(\cdot))^\alpha$. We define furthermore

$$Z_\delta := \exp \left(-\frac{1}{2} \delta^2 \|\sigma_L - \theta'\|^2 T + \delta (\sigma_L - \theta') W_T \right),$$

which has the cumulative distribution function

$$F_{Z_\delta}(z) = \mathcal{N} \left(\frac{\ln z + \frac{1}{2} \delta^2 \|\sigma_L - \theta'\|^2 T}{\delta \|\sigma_L - \theta'\| \sqrt{T}} \right)$$

and denote its corresponding density by f_{Z_δ} . The quantile function is given by

$$F_{Z_\delta}^{\leftarrow}(p) = \exp \left(-\frac{1}{2} \delta^2 \|\sigma_L - \theta'\|^2 T + \delta \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right)$$

¹For $\nu \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R}^+$ and $Y \sim \mathcal{LN}(\mu, \sigma^2)$, it holds

$$\mathbb{E}[Y^\nu \mathbf{1}_{Y \in (c_1, c_2)}] = \exp \left(\nu \mu + \frac{1}{2} \nu^2 \sigma^2 \right) \left(\mathcal{N} \left(\frac{\ln c_2 - \mu - \nu \sigma^2}{\sigma} \right) - \mathcal{N} \left(\frac{\ln c_1 - \mu - \nu \sigma^2}{\sigma} \right) \right).$$

and we define

$$\begin{aligned}\bar{w}_\delta(p) &:= \int_0^{F_{Z_\delta}^{\leftarrow}(p)} r f_{Z_\delta}(r) dr \\ &= \mathbb{E} \left[Z_\delta \mathbf{1}_{Z_\delta \leq F_{Z_\delta}^{\leftarrow}(p)} \right] = \mathcal{N} \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right).\end{aligned}$$

Thus, we have

$$w(p) = (\bar{w}_\delta(p))^\alpha.$$

Lemma 4.1. *For $\delta \in (0, 1]$ and $\alpha \in (0, 1]$, the distortion function w satisfies (M) and the first derivative is given by*

$$\begin{aligned}w'(p) &= \alpha \mathcal{N}^{\alpha-1} \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right) F_Z^{\leftarrow}(p) \\ &\quad \cdot \exp \left(\frac{1}{2} (1 - \delta^2) \|\sigma_L - \theta'\|^2 T + (\delta - 1) \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right).\end{aligned}$$

Furthermore, w is reverse S -shaped for $\alpha \in (0, 1)$.

Proof. See Appendix C. □

Figure 1 shows the probability distortion function for varying parameters. From the definition of $\mathbb{V}_+$ and $\mathbb{V}_-$, we have for a random variable Y with distribution function F_Y and density function f_Y

$$\mathbb{E} [U_+(Y) w'_+(1 - F_Y(y))] = \int_{-\infty}^{\infty} U_+(y) w'_+(1 - F_Y(y)) f_Y(y) dy.$$

This can be interpreted as an expectation of $U_+(Y)$ under a distorted probability measure with the distorted density function of Y given by

$$f_Y^w(y) := w'_+(1 - F_Y(y)) f_Y(y)$$

and the corresponding distribution function given by

$$F_Y^w(y) := \int_{-\infty}^y f_Y^w(s) ds = 1 - w_+(1 - F_Y(y)).$$

Figure 2 shows a plot of f_Y^w , with Y being standard normally distributed for illustrative purposes. The influence of different parameter values can be seen clearly: for a fixed α , different values for δ result in a simple shift. When interpreting Y as the return of a portfolio or considering the funding ratio $\bar{X}$, an increase in δ leads to a heavier lower tail of the distribution and can be interpreted as an increase of the downside risk. In case $\alpha = 1$, this corresponds to an increase in downside risk only. A decrease in the parameter α then yields to an increase in the upper tail of the distribution. By combining both effects, the probability for events from both tails is increased by the distortion. This is illustrated with the parameter set $\alpha = 0.5$, $\delta = 1$.

Figure 1. The probability distortion function w . Individual parameters are varied as indicated in the legend.

Figure 2. The density of a standard normal random variable that was distorted by the probability distortion function. Individual parameters are varied as indicated in the legend.

Optimal Behavior in the Well-Funded Case

We now assume $\delta_{\pm} \in (0, 1]$, $\alpha_{\pm} \in (0, 1]$ as well as $(\mathbf{K})$. Note that $(\mathbf{M})$ always holds due to Lemma 4.1 and that the distortion on the losses is especially needed to ensure that the problem is well-posed. Then we derive the optimal strategy in the well-funded case when using w from above.

Theorem 4.2 (Three-Fund Separation, CPT, $\frac{A_0}{L_0} \geq \mathbf{B}$). *Let $(\mathbf{K})$ be satisfied and $\frac{A_0}{L_0} \geq \mathbf{B}$. The optimal terminal funding ratio is then given by*

$$X^* - B = \bar{X}^* = \frac{\frac{A_0}{L_0} - B}{G(\infty)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T\right)}{\alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + (\frac{1}{2} - \delta_+) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \right)^{\frac{1}{\gamma-1}}$$

and the optimal investment strategy is given by

$$\varphi^*(t) = \lambda^{CPT}((\bar{X}^*)^+(t, Z_t), B) \varphi^{PS} + (1 - \lambda^{CPT}((\bar{X}^*)^+(t, Z_t), B)) \varphi^{LH},$$

with

$$\lambda^{CPT}((\bar{X}^*)^+(t, Z_t), B) = \frac{-Z_t \frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t)}{(\bar{X}^*)^+(t, Z_t) + B},$$

where the funding ratio at time t corresponding to $(\bar{X}^*)^+$ is denoted by

$$(\bar{X}^*)^+(t, Z_t) := \mathbb{E}_{\mathbb{Q}^L} [(\bar{X}^*)^+ | \mathcal{F}_t]$$

and $\frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t)$ is the derivative of $(\bar{X}^*)^+(t, Z_t)$ with respect to the second component.

Proof. Using Lemma 4.1, we have

$$\begin{aligned} w'_+(F_Z(Z)) &= \alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} - \delta_+ \|\sigma_L - \theta'\| \sqrt{T} \right) \cdot \\ &\quad \exp \left(-\frac{1}{2} \delta_+^2 \|\sigma_L - \theta'\|^2 T + \delta_+ \|\sigma_L - \theta'\| \sqrt{T} \frac{\ln Z + \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \\ &= \alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + (\frac{1}{2} - \delta_+) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \cdot \\ &\quad \exp \left(\delta_+ \ln Z - \frac{1}{2} (\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T \right) \\ &= \alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + (\frac{1}{2} - \delta_+) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \cdot \\ &\quad Z^{\delta_+} \exp \left(-\frac{1}{2} (\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T \right). \end{aligned}$$

Hence, using Theorem 4.1 for the well-funded case, $X^* = (\bar{X}^*)^+ + B$, with

$$\begin{aligned} (\bar{X}^*)^+ &= \frac{\frac{A_0}{L_0} - B}{G(\infty)} \left(\frac{Z}{w'_+(F_Z(Z))} \right)^{\frac{1}{\gamma-1}} \\ &= \frac{\frac{A_0}{L_0} - B}{G(\infty)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T\right)}{\alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + (\frac{1}{2}-\delta_+) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \right)^{\frac{1}{\gamma-1}}. \end{aligned}$$

We use Appendix D with $(\bar{X}^*)^-(t, Z_t) = 0$ to receive the optimal investment strategy for $X^*(t, Z_t) = (\bar{X}^*)^+(t, Z_t) + B$:

$$\begin{aligned} \varphi^*(t) &= \frac{1}{X^*(t, Z_t)} \left(Z_t \frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t) (\varphi^{LH} - \varphi^{PS}) + ((\bar{X}^*)^+(t, Z_t) + B) \varphi^{LH} \right) \\ &= \frac{-Z_t \frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t)}{(\bar{X}^*)^+(t, Z_t) + B} \varphi^{PS} + \left(1 - \frac{-Z_t \frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t)}{(\bar{X}^*)^+(t, Z_t) + B} \right) \varphi^{LH}. \end{aligned}$$

□

Due to this optimal payoff with $(\bar{X}^*)^- = 0$, the funding ratio never falls below B . We have again a three-fund separation. In order to get a more explicit result for λ^{CPT} , we consider the case $\alpha_+ = 1$ in the following corollary.

Corollary 4.1 (Three-Fund Separation, CPT, $\frac{A_0}{L_0} \geq B$, $\alpha_+ = 1$).

Let (K) be satisfied, $\frac{A_0}{L_0} \geq B$ and $\alpha_+ = 1$. The optimal terminal funding ratio is then given by

$$\begin{aligned} X^* - B &= \bar{X}^* = (\bar{X}^*)^+ \\ &= \left(\frac{A_0}{L_0} - B \right) \exp \left(\frac{1}{2} \frac{(1 - \delta_+)(\delta_+ - \gamma)}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T \right) Z^{\frac{1-\delta_+}{\gamma-1}} \end{aligned}$$

and the optimal investment strategy is given by

$$\varphi^*(t) = \lambda^{CPT}(X_t^\varphi, B) \varphi^{PS} + (1 - \lambda^{CPT}(X_t^\varphi, B)) \varphi^{LH},$$

with

$$\lambda^{CPT}(X_t^\varphi, B) = \frac{X_t^\varphi - B}{X_t^\varphi} \cdot \frac{1 - \delta_+}{1 - \gamma}.$$

Proof. For $\alpha_+ = 1$, we have with the proof of Theorem 4.2

$$w'_+(F_Z(Z)) = Z^{\delta_+} \exp \left(-\frac{1}{2} (\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T \right).$$

Thus, we have with Footnote 1 and with $\nu = \frac{\delta_+ - \gamma}{1 - \gamma}$, $c_2 = c$, and $c_1 = 0$

$$\begin{aligned} G(c) &= \mathbb{E} \left[Z^{\frac{\gamma}{\gamma-1}} w'_+ (F_Z(Z))^{\frac{1}{1-\gamma}} \mathbf{1}_{Z \leq c} \right] \\ &= \exp \left(-\frac{1}{2} \frac{\delta_+^2 - \delta_+}{1 - \gamma} \|\sigma_L - \theta'\|^2 T \right) \mathbb{E} \left[Z^{\frac{\delta_+ - \gamma}{1 - \gamma}} \mathbf{1}_{Z \leq c} \right] \\ &= \exp \left(\frac{1}{2} \left(\left(\frac{\delta_+ - \gamma}{1 - \gamma} \right)^2 - \frac{\delta_+ - \gamma}{1 - \gamma} - \frac{\delta_+^2 - \delta_+}{1 - \gamma} \right) \|\sigma_L - \theta'\|^2 T \right) \\ &\quad \mathcal{N} \left(\frac{\ln c + \left(\frac{1}{2} - \frac{\delta_+ - \gamma}{1 - \gamma} \right) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \\ &= \exp \left(\frac{1}{2} \frac{(\delta_+ - 1)^2 \gamma}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T \right) \mathcal{N} \left(\frac{\ln c + \left(\frac{1}{2} - \frac{\delta_+ - \gamma}{1 - \gamma} \right) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right). \end{aligned}$$

Hence, in the well-funded case, the optimal funding ratio is with Theorem 4.2

$$\begin{aligned} (\bar{X}^*)^+ &= \frac{\frac{A_0}{L_0} - B}{G(\infty)} \left(Z^{1 - \delta_+} \exp \left(\frac{1}{2} (\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T \right) \right)^{\frac{1}{\gamma-1}} \\ &= \left(\frac{A_0}{L_0} - B \right) \exp \left(-\frac{1}{2} \frac{(\delta_+ - 1)^2 \gamma}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T \right) \\ &\quad Z^{\frac{1 - \delta_+}{\gamma-1}} \exp \left(\frac{1}{2} \frac{\delta_+^2 - \delta_+}{\gamma - 1} \|\sigma_L - \theta'\|^2 T \right) \\ &= \left(\frac{A_0}{L_0} - B \right) \exp \left(\frac{1}{2} \frac{(1 - \delta_+)(\delta_+ - \gamma)}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T \right) Z^{\frac{1 - \delta_+}{\gamma-1}}. \end{aligned}$$

We first compute the value of $(\bar{X}^*)^+$ at time t , denoted by $(\bar{X}^*)^+(t, Z_t)$ using Footnote 1 with $\nu = \frac{1 - \delta_+}{\gamma - 1}$, $c_2 = \infty$, and $c_1 = 0$:

$$\begin{aligned} (\bar{X}^*)^+(t, Z_t) &= \left(\frac{A_0}{L_0} - B \right) \exp \left(\frac{1}{2} \frac{(1 - \delta_+)(\delta_+ - \gamma)}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T \right) Z_t^{\frac{1 - \delta_+}{\gamma-1}} \\ &\quad \exp \left(\frac{1}{2} \left(\frac{1 - \delta_+}{\gamma - 1} + 1 \right) \frac{1 - \delta_+}{\gamma - 1} \|\sigma_L - \theta'\|^2 (T - t) \right) \\ &= \left(\frac{A_0}{L_0} - B \right) Z_t^{\frac{1 - \delta_+}{\gamma-1}} \exp \left(-\frac{1}{2} \left(\frac{1 - \delta_+}{\gamma - 1} + 1 \right) \frac{1 - \delta_+}{\gamma - 1} \|\sigma_L - \theta'\|^2 t \right). \end{aligned}$$

We know from Appendix D that the corresponding replicating strategy can be written as

$$\varphi_+(t) = \varphi^{LH} + \frac{1 - \delta_+}{1 - \gamma} (\varphi^{PS} - \varphi^{LH})$$

and since $(\bar{X}^*)^-(t, Z_t) = 0$, the optimal investment strategy for $X^*(t, Z_t) = (\bar{X}^*)^+(t, Z_t) + B$ is given by

$$\begin{aligned}\varphi^*(t) &= \frac{1}{(\bar{X}^*)^+(t, Z_t) + B} ((\bar{X}^*)^+(t, Z_t)\varphi_+(t) + B\varphi^{LH}) \\ &= \varphi^{LH} + \frac{(\bar{X}^*)^+(t, Z_t)}{(\bar{X}^*)^+(t, Z_t) + B} \cdot \frac{1 - \delta_+}{1 - \gamma} (\varphi^{PS} - \varphi^{LH}) \\ &= \varphi^{LH} + \frac{X^*(t, Z_t) - B}{X^*(t, Z_t)} \cdot \frac{1 - \delta_+}{1 - \gamma} (\varphi^{PS} - \varphi^{LH}).\end{aligned}\quad \square$$

The optimal strategy results again in the known separation into performance seeking and liability hedging portfolio with the remaining amount being invested in the risk-free asset. Note that the relative risk aversion of the power utility function, $1 - \gamma$, is positive.

Therefore, we identify $\frac{1 - \delta_+}{1 - \gamma}$ as investor-specific pre-factor that can be seen as a measure for the risk-appetite of the investor. An increasing δ_+ shifts the density to the left and therefore tempers the influence of the risk-preference of the investor in the strategy, as it makes $\frac{1 - \delta_+}{1 - \gamma}$ smaller and thus the strategy more cautious in general. In particular, the performance seeking part of the optimal investment strategy corresponds to a CPPI-strategy with multiplier $m := \frac{1 - \delta_+}{1 - \gamma}$ and this ensures that the funding ratio never falls below B .

Optimal Behavior in the Underfunded Case

We turn to the underfunded case. If $c^* = 0$ is the only solution to the optimization problem (C), then Theorem 4.1 states that the optimal funding ratio is

$$X^* = \frac{A_0}{L_0}.$$

Therefore, the optimal strategy is to hedge the liabilities perfectly with

$$\varphi^{LH} = (\sigma')^{-1} \sigma'_L$$

to keep the funding ratio constant and thus to preserve the current level of funding and not risk an even lower funding for potential profit. Remember that $Z \leq c^* \Leftrightarrow X^* \geq B$. So if $c^* = 0$, then

$$\mathbb{P}(X^* \geq B) = \mathbb{P}(Z \leq 0) = 0.$$

This means that there is zero probability that we will achieve funded or well-funded status and thus no reason to risk anything and try. Therefore, we only consider the case $c^* > 0$ in the following results.

Proposition 4.1 (Optimal Payoff, CPT, $\frac{A_0}{L_0} < B$). Let $\inf_{c>0} k(c) > 1$ be satisfied and $\frac{A_0}{L_0} < 0$. If (C) admits an optimal solution $c^* > 0$, the optimal terminal funding ratio is given by

$$X^* - B = \bar{X}^* = (\bar{X}^*)^+ - (\bar{X}^*)^-,$$

with

$$\begin{aligned} (\bar{X}^*)^+ &= \frac{v_+^*}{G(c^*)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T\right)}{\alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + (\frac{1}{2} - \delta_+) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \right)^{\frac{1}{\gamma-1}} \mathbf{1}_{Z \leq c^*} \\ (\bar{X}^*)^- &= \frac{v_+^* - \frac{A_0}{L_0} + B}{1 - \mathcal{N} \left(\frac{\ln c^* - \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \mathbf{1}_{Z > c^*} \end{aligned}$$

and

$$v_+^* = \frac{B - \frac{A_0}{L_0}}{k(c^*)^{1/(1-\gamma)} - 1}.$$

Proof. Proceeding as in the proof of Theorem 4.2, we find with Theorem 4.1

$$\begin{aligned} (\bar{X}^*)^+ &= \frac{v_+^*}{G(c^*)} \left(\frac{Z}{w'_+(F_Z(Z))} \right)^{\frac{1}{\gamma-1}} \mathbf{1}_{Z \leq c^*} \\ &= \frac{v_+^*}{G(c^*)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T\right)}{\alpha_+ \mathcal{N}^{\alpha_+-1} \left(\frac{\ln Z + (\frac{1}{2} - \delta_+) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \right)^{\frac{1}{\gamma-1}} \mathbf{1}_{Z \leq c^*}. \end{aligned}$$

The second pay-off is given by

$$(\bar{X}^*)^- = \frac{v_+^* - \frac{A_0}{L_0} + B}{\mathbb{E}[Z \mathbf{1}_{Z > c^*}]} \mathbf{1}_{Z > c^*} = \frac{v_+^* - \frac{A_0}{L_0} + B}{1 - \mathcal{N} \left(\frac{\ln c^* - \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \mathbf{1}_{Z > c^*}.$$

□

In contrast to the well-funded case, in which the CPPI-part for the performance seeking portfolio implies that there is no risk of falling below the reference point B , the optimal strategy in the underfunded case corresponds to a leveraged strategy and thus includes more risk. The reason for this difference is that in the well-funded case, the risk-seeking area of the convex utility function below the reference point is never reached, whereas

this area has substantial influence in the case of initial underfunding. The investor tries to achieve a terminal wealth above B by taking additional risk. In case the market evolves in an advantageous way, in particular if $Z \leq c^*$, then the investor receives the payoff $(\bar{X}^*)^+ + B$. In the other case, i.e. $Z > c^*$, the investor suffers a constant loss with terminal wealth $B - (\bar{X}^*)^-$ due to the additional risk. For $\alpha_+ = 1$, the following result states the optimal investment strategy in closed form.

Theorem 4.3 (*Three-Fund Separation, CPT, $\frac{A_0}{L_0} < B$, $\alpha_+ = 1$*). *Let $\inf_{c>0} k(c) > 1$ be satisfied, $\frac{A_0}{L_0} < 0$ and $\alpha_+ = 1$. If (C) admits an optimal solution $c^* > 0$, the optimal funding ratio is given by*

$$X^* - B = \bar{X}^* = (\bar{X}^*)^+ - (\bar{X}^*)^-,$$

with

$$\begin{aligned} (\bar{X}^*)^+ &= \frac{v_+^*}{\mathcal{N}\left(\frac{\ln c^* + (\frac{1}{2} - \frac{\delta_+ - \gamma}{1 - \gamma})\|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\|\sqrt{T}}\right)} \\ &\quad \exp\left(\frac{1}{2} \frac{(1 - \delta_+)(\delta_+ - \gamma)}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T\right) Z^{\frac{1 - \delta_+}{\gamma - 1}} \mathbf{1}_{Z \leq c^*}, \\ (\bar{X}^*)^- &= \frac{v_+^* - \frac{A_0}{L_0} + 1}{1 - \mathcal{N}\left(\frac{\ln c^* - \frac{1}{2}\|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\|\sqrt{T}}\right)} \mathbf{1}_{Z > c^*} \end{aligned}$$

and

$$v_+^* = \frac{B - \frac{A_0}{L_0}}{k(c^*)^{1/(1-\gamma)} - 1}.$$

The optimal investment strategy is given by

$$\varphi^*(t) = \lambda^{CPT} \cdot \varphi^{PS} + (1 - \lambda^{CPT}) \cdot \varphi^{LH},$$

with

$$\begin{aligned}\lambda^{CPT} &= \frac{1}{X_t^\varphi} \cdot (X_+^\varphi(t) \cdot \lambda_+ - X_-^\varphi(t) \cdot \lambda_-) \\ \lambda_+ &= \left(\frac{1 - \delta_+}{1 - \gamma} + \frac{1}{\|\sigma_L - \theta'\| \sqrt{T - t}} \cdot \frac{\mathcal{N}'\left(d\left(\frac{c^*}{Z_t}, T - t, \frac{1 - \delta_+}{\gamma - 1}\right)\right)}{\mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T - t, \frac{1 - \delta_+}{\gamma - 1}\right)\right)} \right), \\ \lambda_- &= \frac{\mathcal{N}'\left(d\left(\frac{c^*}{Z_t}, T - t, 0\right)\right)}{1 - \mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T - t, 0\right)\right)} \cdot \frac{1}{\|\sigma'_L - \theta\| \sqrt{T - t}}, \\ X_+^\varphi(t) &= \frac{v_+^* Z_t^{\frac{1 - \delta_+}{\gamma - 1}} \cdot \mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T - t, \frac{1 - \delta_+}{\gamma - 1}\right)\right)}{\mathcal{N}\left(\frac{\ln c^* + (-\frac{1}{2} \cdot \frac{1 - \delta_+}{\gamma - 1}) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}}\right)} \\ &\quad \cdot \exp\left(-\frac{1}{2} \left(\frac{1 - \delta_+}{\gamma - 1} + 1\right) \cdot \frac{1 - \delta_+}{\gamma - 1} \|\sigma_L - \theta'\|^2 t\right), \\ X_-^\varphi(t) &= \frac{v_+^* - \frac{A_0}{L_0} + B}{1 - \mathcal{N}\left(\frac{\ln c^* - \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}}\right)} \cdot \left(1 - \mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T - t, 0\right)\right)\right) \\ \text{and } d(c, s, v) &:= \frac{\ln(c) - (v + \frac{1}{2}) \|\sigma_L - \theta'\|^2 s}{\|\sigma_L - \theta'\| \sqrt{s}}.\end{aligned}$$

Proof. From the proof of Corollary 4.1, we know that

$$\begin{aligned}G(c^*) &= \exp\left(\frac{1}{2} \frac{(\delta_+ - 1)^2 \gamma}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T\right) \\ &\quad \cdot \mathcal{N}\left(\frac{\ln c^* + (\frac{1}{2} - \frac{\delta_+ - \gamma}{1 - \gamma}) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}}\right).\end{aligned}$$

Applying Proposition 4.1 yields

$$\begin{aligned}(\bar{X}^*)^+ &= \frac{v_+^*}{G(c^*)} \left(Z^{1 - \delta_+} \exp\left(\frac{1}{2} (\delta_+^2 - \delta_+) \|\sigma_L - \theta'\|^2 T\right) \right)^{\frac{1}{\gamma - 1}} \mathbf{1}_{Z \leq c^*} \\ &= \frac{v_+^*}{\mathcal{N}\left(\frac{\ln c^* + (\frac{1}{2} - \frac{\delta_+ - \gamma}{1 - \gamma}) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}}\right)} \\ &\quad \exp\left(\frac{1}{2} \frac{(1 - \delta_+)(\delta_+ - \gamma)}{(1 - \gamma)^2} \|\sigma_L - \theta'\|^2 T\right) Z^{\frac{1 - \delta_+}{\gamma - 1}} \mathbf{1}_{Z \leq c^*}.\end{aligned}$$

We can apply Appendix D to derive

$$(\bar{X}^*)^+(t, Z_t) = \frac{v_+^* Z_t^{\frac{1-\delta_+}{\gamma-1}}}{\mathcal{N}\left(\frac{\ln c^* + (\frac{1}{2} - \frac{\delta_+ - \gamma}{1-\gamma}) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}}}\right)} \mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T-t, \frac{1-\delta_+}{\gamma-1}\right)\right) \cdot \exp\left(-\frac{1}{2}\left(\frac{1-\delta_+}{\gamma-1} + 1\right) \frac{1-\delta_+}{\gamma-1} \|\sigma_L - \theta'\|^2 t\right)$$

and the replicating strategy is

$$\varphi_+(t) = \varphi^{LH} + \left(\frac{1-\delta_+}{1-\gamma} + \frac{1}{\|\sigma_L - \theta'\| \sqrt{T-t}} \frac{\mathcal{N}'\left(d\left(\frac{c^*}{Z_t}, T-t, \frac{1-\delta_+}{\gamma-1}\right)\right)}{\mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T-t, \frac{1-\delta_+}{\gamma-1}\right)\right)} \right) \cdot (\varphi^{PS} - \varphi^{LH}).$$

The second pay-off is given by Proposition 4.1 and with Appendix D, $c_2 = \infty$, $c_1 = c^*$ and $\nu = 0$, we have

$$(\bar{X}^*)^-(t, Z_t) = \frac{v_+^* - \frac{A_0}{L_0} + B}{1 - \mathcal{N}\left(\frac{\ln c^* - \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}}}\right)} \left(1 - \mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T-t, 0\right)\right)\right).$$

Moreover, the replicating strategy φ_- of $(\bar{X}^*)^-$ is

$$\varphi_- = \varphi^{LH} - \frac{\mathcal{N}'\left(d\left(\frac{c^*}{Z_t}, T-t, 0\right)\right)}{1 - \mathcal{N}\left(d\left(\frac{c^*}{Z_t}, T-t, 0\right)\right)} \frac{1}{\|\sigma'_L - \theta\| \sqrt{T-t}} (\varphi^{PS} - \varphi^{LH}).$$

This allows us to compute the replicating strategy of the optimal funding ratio as

$$\varphi^*(t) = \frac{1}{X^*(t, Z_t)} ((\bar{X}^*)^+(t, Z_t) \varphi_+(t) - (\bar{X}^*)^-(t, Z_t) \varphi_-(t) + B \varphi^{LH})$$

and to represent it as in the statement of the theorem by rearranging the terms and with $X^*(t, Z_t) = (\bar{X}^*)^+(t, Z_t) - (\bar{X}^*)^-(t, Z_t) + B$. $\square$

5. Comparison

5.1. Partial surplus optimization

After the derivation of the optimal strategies, we now show that the CPT approach is a generalization of the funding ratio optimization as well as a continuous-time analogue of the partial surplus optimization. This also

establishes the link between partial surplus optimization in discrete time and funding ratio optimization in continuous time.² The partial surplus in [1] is defined as $A_T^\varphi - kL_T$, $k \in [0, 1]$ and the surplus return introduced as

$$S_T^\varphi := \frac{A_T^\varphi - kL_T}{A_0} = r_{A^\varphi}(T) - k \frac{L_0}{A_0} r_L(T),$$

with $r_{A^\varphi}(T)$ being the performance of the assets and $r_L(T)$ being the performance of the liabilities from $t = 0$ until $t = T$.

The authors of [1] apply a mean-variance utility to the partial surplus

$$\max_{\varphi} \mathbb{E}[U(S_T^\varphi)] = \max_{\varphi} [\mathbb{E}[S_T^\varphi] - \gamma_{MV} \text{Var}(S_T^\varphi)],$$

with risk aversion parameter $\gamma_{MV} > 0$. They show that this problem can also be written as

$$\max_{\varphi} \left[\mathbb{E}[r_A^\varphi(T)] - \gamma_{MV} \text{Var}[r_A^\varphi(T)] + \frac{2\gamma_{MV} k L_0}{A_0} \text{Cov}[r_A^\varphi(T), r_L(T)] \right]. \quad (\text{MV})$$

We consider the logarithmic returns

$$r_{A^\varphi}(T) := \ln \left(\frac{A_T^\varphi}{A_0} \right) \quad \text{and} \quad r_L(T) := \ln \left(\frac{L_T}{L_0} \right).$$

For the market model from the previous sections, we calculate an optimal static investment strategy without allowance for intertemporal portfolio choice.

Theorem 5.1 (Three-Fund Separation, Mean-Variance Utility).

The optimal investment strategy for (MV) is given by

$$\varphi_{\text{Sharpe}}^* = \lambda^S \varphi^{PS} + (1 - \lambda^S) \frac{kL_0}{A_0} \varphi^{LH},$$

with $\lambda^S = \frac{1}{2\gamma_{MV} + 1}$. The remaining wealth $1 - \mathbf{1}' \varphi_{\text{Sharpe}}^*$ is invested in the risk-free asset.

Proof. Setting the gradient of the expected utility equal to zero with respect to φ yields the optimal solution. It is a maximum as the Hessian with respect to φ is negative definite due to $\gamma_{MV} > 0$. □

²In contrast to this funding ratio optimization, the intertemporal surplus management approach by [5] cannot be interpreted as a generalization of the funding ratio optimization.

5.2. Connection between CPT optimization, funding ratio optimization and partial surplus optimization

After having obtained three-fund separations for the funding ratio optimization in an expected utility framework and in a CPT framework as well as for the partial surplus optimization, we compare the results from Corollary 3.1, Corollary 4.1 and Theorem 5.1. While the optimal investment strategy depends on the initial funding ratio for the static mean-variance approach, the optimal investment strategy for the CPT approach with $\frac{A_0}{L_0} \geq B$ depends dynamically on X_t^φ and the solution for the expected utility approach with power utility is independent of the funding ratio at any time. In Figure 3 and Figure 4, the wealth allocated in the performance seeking portfolio and the liability hedging portfolio at $t = 0$ is illustrated. Unless otherwise stated, the parameters chosen are $\gamma_{MV} = 5$, $k = 0.9$ and $\frac{A_0}{L_0} = 1.1$ for the mean-variance utility approach, $\gamma = -10$ for the expected utility approach with power utility and $B = 0.5$, $\delta_+ = 0.5$ as well as also $\frac{A_0}{L_0} = 1.1$ for the CPT approach. Since we want to consider downside risk only, we set $\alpha = 1$

Figure 3. Weights of the performance seeking portfolio in the different settings.

in the CPT example. If we set $B = 0$, the CPT approach for $\frac{A_0}{L_0} \geq B$ corresponds to a pure CPT funding ratio optimization. The optimal strategy differs from the expected utility approach only by the factor $1 - \delta_+$ caused by the probability distortion. Figures 5 and 6 show that the allocations in the performance seeking portfolio and liability hedging portfolio for the CPT approach converge to the optimal investment strategy in the expected utility approach with power utility when $B \rightarrow 0$ and $\delta_+ \rightarrow 0$. Thus, the CPT approach is a generalization of the presented expected utility theory approach. On the other hand, CPT is also a generalization of the partial surplus optimization. In the CPT context we optimize the utility of $\bar{X} = X_T^\varphi - B = \frac{A_T^\varphi}{L_T} - B$ and we have

$$\frac{A_T^\varphi}{L_T} - B \geq 0 \Leftrightarrow A_T^\varphi \geq BL_T \Leftrightarrow A_T^\varphi - BL_T \geq 0.$$

Hence, the reference point B , indicating to which extent the liabilities are considered, corresponds to k in the partial surplus optimization.

Figure 4. Weights of the liability hedging portfolio in the different settings.

Figure 5. Weights of the performance seeking portfolio in the different settings.

Figure 6. Weights of the liability hedging portfolio in the different settings.

6. Conclusion

Using the market model from [2] for the assets and liabilities, we derive optimal investment strategies in a CPT funding ratio optimization framework with S-shaped utility function and a distortion which has only few parameters but still the desired properties. The optimal investment strategies with initial funding level above and below the reference point differ, but can be represented as a three-fund theorem in each of the two cases. We compare the results from the CPT optimization to the results from [2] obtained for the funding ratio optimization in an expected utility framework and to an application of the partial surplus approach in [1] to our market model. We conclude that the CPT approach can be considered as a generalization of both, the partial surplus approach and the funding ratio optimization.

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Appendix A. Solution of the HJB equation

We use the well-known separation approach

$$\Phi(t, X_t^\varphi) = U(X_t^\varphi) \Delta(t), \quad (\text{A.1})$$

where $\Delta : [0, T] \rightarrow \mathbb{R}_+$. The function Φ is strictly concave in X_t^φ as U is strictly concave and thus we can find a unique maximizer φ^* . As the relative risk aversion of U is constant $1 - \gamma$, the relative risk tolerance is

$$\lambda^{EUP}(X_t^\varphi, t) = \frac{1}{1 - \gamma}.$$

The resulting maximizer is

$$\varphi^* = \left(1 - \frac{1}{1 - \gamma}\right) \varphi^{LH} + \frac{1}{1 - \gamma} \varphi^{PS},$$

the investment in the risk-free asset is $\varphi_0^* = 1 - \mathbf{1}' \varphi^*$. Using the optimal strategy in $\mu_X^\varphi(t)$ and $\sigma_X^\varphi(t)$ results in

$$\begin{aligned} \mu_X^{\varphi^*}(t) &= r + \varphi_t^{*'}(\mu - r\mathbf{1}) - \mu_L + \sigma_L \sigma_L' + \sigma_\epsilon^2 - \sigma_L \sigma' \varphi_t^* \\ &= r - \mu_L + \frac{1}{1 - \gamma} \sigma_L \sigma_L' + \sigma_\epsilon^2 + \frac{1}{1 - \gamma} \theta' \theta + \left(1 - \frac{2}{1 - \gamma}\right) \sigma_L \theta. \end{aligned}$$

and

$$\begin{aligned}\sigma_X^{\varphi^*}(t)^2 &= (\varphi_t^{*'}\sigma - \sigma_L) (\varphi_t^{*'}\sigma - \sigma_L)' + \sigma_\epsilon^2 \\ &= \frac{1}{(1-\gamma)^2} \sigma_L \sigma_L' + 2 \frac{1}{1-\gamma} (1 - \frac{1}{1-\gamma}) \sigma_L \theta + \frac{1}{(1-\gamma)^2} \theta' \theta + \sigma_\epsilon^2.\end{aligned}$$

The assumption on the form of the solution (A.1) and the above results reduces the HJB equation to the ODE

$$0 = \Delta'(t) + \left[\mu_X(\varphi^*, t) \gamma + \frac{1}{2} (\sigma_X(\varphi^*, t))^2 \gamma (1 - \gamma) \right] \Delta(t).$$

Due to the terminal condition $\Delta(T) = 1$, it has the solution³

$$\Delta(t) = \exp \left(\left[\mu_X(\varphi^*, t) \gamma + \frac{1}{2} (\sigma_X(\varphi^*, t))^2 \gamma (1 - \gamma) \right] (T - t) \right).$$

Appendix B. Quantile optimization approach

The optimization approach works by separating the Original Problem (**OP**) into further optimization problems, one for the gains and one for the losses, and a third one that glues them back together. To give an intuition, note that for any feasible solution $\bar{X}$ we have $\bar{X} = (\bar{X})^+ - (\bar{X})^-$ and $(\bar{X})^+$ only influences $\mathbb{V}_+$. In addition, it has a price $\mathbb{E}[Z(\bar{X})^+] =: v_+$ and it only has positive value on an implicitly defined set $A := \{\bar{X} \geq 0\}$. We will call the problem of optimizing $(\bar{X})^+$ the *Gains Problem* (**GP**)

$$\begin{aligned}\sup_{\bar{X}} \quad & \mathbb{V}_+((\bar{X})^+) \\ \text{s.t.} \quad & \mathbb{E}[Z(\bar{X})^+] = v_+, \\ & \bar{X} \text{ is } \mathcal{F}_T\text{-measurable and lower-bounded.}\end{aligned} \tag{GP}$$

The other part of the solution $(\bar{X})^-$ only influences $\mathbb{V}_-$, exists on the complement $A^c = \{\bar{X} < 0\}$ and has a price $\mathbb{E}[Z(\bar{X})^-] := v_+ - \left(\frac{A_0}{L_0} - B \right)$.

³It should be noted that this is a heuristic derivation. For mathematical rigorosity, a verification theorem would have to be shown, which is beyond the scope of this paper. However, for $\gamma \in (0, 1)$, a verification theorem which is directly applicable can be found in [17], Theorem 3.5.2.

The corresponding *Loss Problem* (**LP**) is given by

$$\begin{aligned} \inf_{\bar{X}} \quad & \mathbb{V}_-((\bar{X})^-) \\ \text{s.t.} \quad & \mathbb{E}[Z(\bar{X})^-] = v_+ - \left(\frac{A_0}{L_0} - B \right) \end{aligned} \quad (\mathbf{LP})$$

$\bar{X}$ is $\mathcal{F}_T$ -measurable and upper-bounded.

The link between (**GP**) and (**LP**) is the so-called *Gluing Problem*. For this, note that any solutions of both (**GP**) and (**LP**) depend on two common parameters: A and v_+ . If any solution depends on those parameters, then so do the optimal solutions. Knowing that the optimal values of the gains problem $\Phi_+(v_+, A)$ and loss problem $\Phi_-(v_+, A)$ depend on v_+ and A , the Gluing Problem boils down to finding an optimal pair (v_+, A^*) that maximizes $\Phi_+(v_+, A) - \Phi_-(v_+, A)$. To simplify the Gluing Problem even further, it turns out that an optimal set A^* always has the form $A^* = \{Z \leq c^*\}$, $c \geq 0$ (see [12], Theorem 5.1). Hence, it is enough to look at solutions of the form $\Phi_{\pm}(v_+, \{Z \leq c\}) =: \Phi_{\pm}(v_+, c)$. For an indication on why this is true, remember that $F_Z(Z)$ and consequently $1 - F_Z(Z)$ are uniformly distributed on the unit interval and that $A = \{\bar{X} \geq 0\}$. Then we define

$$c := F_Z^{\leftarrow}(\mathbb{P}(A)) = F_Z^{\leftarrow}(\mathbb{P}(\bar{X} \geq 0)) = F_Z^{\leftarrow}(\mathbb{P}(X \geq B))$$

and see that⁴

$$\begin{aligned} Z \leq c &\Leftrightarrow F_Z(Z) \leq \mathbb{P}(\bar{X} \geq 0) \Leftrightarrow 1 - F_Z(Z) \geq \mathbb{P}(\bar{X} < 0) = F_{\bar{X}}(0) \\ &\Leftrightarrow \bar{X} \stackrel{d}{=} F_{\bar{X}}^{\leftarrow}(1 - F_Z(Z)) \geq 0. \end{aligned}$$

This means that

$$\mathbb{P}(\bar{X} \geq 0) = \mathbb{P}(Z \leq c)$$

for all relevant $\bar{X}$. The Gluing Problem (**GLUE**) is defined as

$$\begin{aligned} \sup_{(v_+, c)} \quad & \Phi_+(v_+, c) - \Phi_-(v_+, c) \\ \text{s.t.} \quad & 0 \leq c \leq \infty. \end{aligned} \quad (\mathbf{GLUE})$$

After solving (**GLUE**), we know the optimal parameters c^* and v_+^* . It was shown that $Z \leq c \Leftrightarrow \bar{X} \geq 0$ in all relevant cases, and thus we have for the

⁴ $\stackrel{d}{=}$ indicates equivalence in distribution. Also note that $\bar{X}$ is continuous.

optimal solutions of the adjusted problems

$$(\bar{X}^*)^+ = \bar{X}^* \mathbf{1}_{\bar{X}^* \geq 0} = \bar{X}^* \mathbf{1}_{Z \leq c^*}$$

and analogously

$$(\bar{X}^*)^- = -\bar{X}^* \mathbf{1}_{\bar{X}^* < 0} = -\bar{X}^* \mathbf{1}_{Z > c^*}.$$

This means that we can also change the optimization problems **(GP)** and **(LP)** to the equivalent problems

$$\begin{aligned} \sup_{\bar{X}} \quad & \mathbb{V}_+(\bar{X} \mathbf{1}_{Z \leq c}) \\ \text{s.t.} \quad & \mathbb{E}[Z \bar{X} \mathbf{1}_{Z \leq c}] = v_+, \\ & \bar{X} \mathbf{1}_{Z \leq c} \geq 0 \end{aligned} \quad (\mathbf{GP}')$$

and

$$\begin{aligned} \inf_{\bar{X}} \quad & \mathbb{V}_-(-\bar{X} \mathbf{1}_{Z > c}) \\ \text{s.t.} \quad & \mathbb{E}[-Z \bar{X} \mathbf{1}_{Z > c}] = v_+ - \left(\frac{A_0}{L_0} - B \right) \\ & -\bar{X} \mathbf{1}_{Z > c} \geq 0. \end{aligned} \quad (\mathbf{LP}')$$

Problems **(GP')** and **(LP')** can then be solved as described in [12].

Appendix C. Probability distortion

Lemma C.1. *The function $\bar{w}_\delta : [0, 1] \rightarrow [0, 1]$ with*

$$\bar{w}_\delta(p) = \mathcal{N} \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right), \quad \delta > 0,$$

is convex.

Proof. We know with

$$\bar{w}_\delta(p) = \int_0^{F_{Z_\delta}^{\leftarrow}(p)} r f_{Z_\delta}(r) dr = \int_0^p F_{Z_\delta}^{\leftarrow}(q) dq$$

that

$$\bar{w}'_\delta(p) = F_{Z_\delta}^{\leftarrow}(p) = \exp \left(-\frac{1}{2} \delta^2 \|\sigma_L - \theta'\|^2 T + \delta \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right).$$

Thus we have

$$\bar{w}''_\delta(p) = \bar{w}'_\delta(p) \frac{\delta \|\sigma_L - \theta'\| \sqrt{T}}{\mathcal{N}'(\mathcal{N}^{\leftarrow}(p))} \geq 0.$$

The derivative of $w(p) = (\bar{w}_\delta(p))^\alpha$ is given by:

$$\begin{aligned}
 w'(p) &= \alpha (\bar{w}_\delta(p))^{\alpha-1} F_{Z_\delta}^{\leftarrow}(p) \\
 &= \alpha \mathcal{N}^{\alpha-1} \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right) \cdot \\
 &\quad \exp \left(-\frac{1}{2} \delta^2 \|\sigma_L - \theta'\|^2 T + \delta \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right) \\
 &= \alpha \mathcal{N}^{\alpha-1} \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right) \\
 &\quad \exp \left(-\frac{1}{2} \|\sigma_L - \theta'\|^2 T + \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right) \cdot \\
 &\quad \exp \left(\frac{1}{2} (1 - \delta^2) \|\sigma_L - \theta'\|^2 T + (\delta - 1) \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right) \\
 &= \alpha \mathcal{N}^{\alpha-1} \left(\mathcal{N}^{\leftarrow}(p) - \delta \|\sigma_L - \theta'\| \sqrt{T} \right) F_Z^{\leftarrow}(p) \cdot \\
 &\quad \exp \left(\frac{1}{2} (1 - \delta^2) \|\sigma_L - \theta'\|^2 T + (\delta - 1) \|\sigma_L - \theta'\| \sqrt{T} \mathcal{N}^{\leftarrow}(p) \right).
 \end{aligned}$$

□

Lemma C.2. For $\delta \in (0, 1]$, $\alpha \in (0, 1]$, the probability distortion function w satisfies the monotonicity condition **(M)**. Moreover, it is reverse-S shaped if in addition $\alpha \in (0, 1)$.

Proof. To show that **(M)** holds, we have to examine

$$\frac{F_Z^{\leftarrow}(p)}{w'(p)}.$$

With

$$c := F_Z^{\leftarrow}(p), \quad I(c) := \left. \frac{w'(p)}{F_Z^{\leftarrow}(p)} \right|_{p=F_Z(c)} = \frac{w'(F_Z(c))}{c}, \quad H(c) := w(F_Z(c)),$$

we get

$$I(c) = \frac{H'(c)}{c F_Z'(c)}$$

and subsequently

$$\begin{aligned}
 I'(c) &= \frac{H''(c) c F_Z'(c) - H'(c) c F_Z''(c) - H_Z'(c) F_Z'(c)}{(c F_Z'(c))^2} \leq 0 \\
 \Leftrightarrow j(c) &:= \frac{c H''(c)}{H'(c)} - \frac{c F_Z''(c)}{F_Z'(c)} = c \left(\frac{H''(c)}{H'(c)} - \frac{F_Z''(c)}{F_Z'(c)} \right) \leq 1.
 \end{aligned}$$

Note that $c > 0$ since Z is log-normally distributed. The following statements are equivalent since F_Z is monotonically increasing:

$$\frac{F_Z^{\leftarrow}(p)}{w'(p)} \text{ is increasing} \Leftrightarrow I(c) \text{ is decreasing} \Leftrightarrow j(c) \leq 1.$$

We have

$$H(c) = w(F_Z(c)) = \mathcal{N}^\alpha \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right).$$

Therefore,

$$H'(c) = \alpha \mathcal{N}^{\alpha-1} \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \mathcal{N}' \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \frac{1}{c \|\sigma_L - \theta'\| \sqrt{T}} > 0$$

and

$$\begin{aligned} H''(c) &= H'(c)(\alpha - 1) \frac{\mathcal{N}' \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)}{\mathcal{N} \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \frac{1}{c \|\sigma_L - \theta'\| \sqrt{T}} \\ &\quad - H'(c) \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \frac{1}{c \|\sigma_L - \theta'\| \sqrt{T}} - H'(c) \frac{1}{c}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{cH''(c)}{H'(c)} &= (\alpha - 1) \frac{\mathcal{N}' \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)}{\mathcal{N} \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right)} \frac{1}{\|\sigma_L - \theta'\| \sqrt{T}} \\ &\quad - \left(\frac{\ln c + (\frac{1}{2} - \delta) \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \frac{1}{\|\sigma_L - \theta'\| \sqrt{T}} - 1. \end{aligned}$$

For the distribution of the kernel (see (4)), it holds

$$F'_Z(c) = \mathcal{N}' \left(\frac{\ln c + \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \frac{1}{c \|\sigma_L - \theta'\| \sqrt{T}}$$

and

$$\begin{aligned} F''_Z(c) &= \mathcal{N}' \left(\frac{\ln c + \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\| \sqrt{T}} \right) \\ &\quad \frac{1}{c \|\sigma_L - \theta'\| \sqrt{T}} \left(- \frac{\ln c + \frac{1}{2} \|\sigma_L - \theta'\|^2 T}{c \|\sigma_L - \theta'\|^2 T} - \frac{1}{c} \right). \end{aligned}$$

Thus the second fraction in j is

$$\frac{cF_Z''(c)}{F_Z'(c)} = -\frac{\ln c + \frac{1}{2}\|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\|^2 T} - 1.$$

Then the function j corresponding to w is

$$\begin{aligned} j(c) &= \frac{\alpha - 1}{\|\sigma_L - \theta'\|\sqrt{T}} \frac{\mathcal{N}'\left(\frac{\ln c + (\frac{1}{2} - \delta)\|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\|\sqrt{T}}\right)}{\mathcal{N}\left(\frac{\ln c + (\frac{1}{2} - \delta)\|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\|\sqrt{T}}\right)} + \delta \\ &= \frac{\alpha - 1}{\|\sigma_L - \theta'\|\sqrt{T}} \frac{\mathcal{N}'(d(c, T, \delta - 1))}{\mathcal{N}(d(c, T, \delta - 1))} + \delta, \end{aligned}$$

with $d(c, T, \delta - 1) := \frac{\ln c + (\frac{1}{2} - \delta)\|\sigma_L - \theta'\|^2 T}{\|\sigma_L - \theta'\|\sqrt{T}}$. Its first term is not positive because $\alpha \in (0, 1]$ and it goes to zero for $c \rightarrow \infty$. Thus we can confirm that $j(c) \leq 1 \Leftrightarrow \delta \leq 1$. This means that w satisfies the monotonicity condition **(M)** if and only if $\delta \leq 1$.

Furthermore, we have

$$H'(c) = w'(F_Z(c))F_Z'(c)$$

and

$$\begin{aligned} H''(c) &= w''(F_Z(c))F_Z'(c)^2 + w'(F_Z(c))F_Z''(c) \\ &= w''(F_Z(c))F_Z'(c)^2 + \frac{H'(c)}{F_Z'(c)}F_Z''(c), \end{aligned}$$

as well as

$$w(p) = H(F_Z^{\leftarrow}(p)) = H(c),$$

and

$$w''(F_Z(c)) = \frac{1}{F_Z'(c)^2} \left[H''(c) - \frac{H'(c)}{F_Z'(c)} F_Z''(c) \right].$$

Finally,

$$\begin{aligned} &w''(F_Z(c)) \\ &< 0 \Leftrightarrow H''(c) - \frac{H'(c)}{F_Z'(c)} F_Z''(c) < 0 \Leftrightarrow c \left(\frac{H''(c)}{H'(c)} - \frac{F_Z''(c)}{F_Z'(c)} \right) = j(c) < 0. \end{aligned}$$

Hence, w is reverse-S shaped if and only if j is first negative and then positive. For this, we assume that $\alpha \in (0, 1)$ and we note that

$$\begin{aligned} j(c) &= (\alpha - 1)c \frac{\partial}{\partial c} (\ln(\mathcal{N}(d(c, T, \delta - 1)))) + \delta \\ &= \delta + \frac{\alpha - 1}{\|\sigma_L - \theta'\|\sqrt{T}} \frac{\partial}{\partial d} (\ln(\mathcal{N}(d)))|_{d=d(c, T, \delta - 1)} \end{aligned}$$

and hence with

$$j(c) = j(d(c)), \quad d(c) := d(c, T, \delta - 1)$$

we have

$$\begin{aligned} j'(c) &= \frac{\partial}{\partial d} j(d)|_{d=d(c)} \cdot \frac{\partial}{\partial c} d(c) \\ &= \frac{\alpha - 1}{\|\sigma_L - \theta'\| \sqrt{T}} \frac{\partial^2}{\partial d^2} (\ln(\mathcal{N}(d)))|_{d=d(c)} \cdot \frac{\partial}{\partial c} d(c, T, \delta - 1). \end{aligned}$$

The factor $\frac{\alpha - 1}{\|\sigma_L - \theta'\| \sqrt{T}}$ is negative because $\alpha \in (0, 1)$. Furthermore, $\frac{\partial}{\partial c} d(c, T, \delta - 1)$ is positive. Thus it remains to be shown that $\frac{\partial^2}{\partial d^2} (\ln(\mathcal{N}(d)))$ is negative. The normal cumulative distribution function is a log-concave function (see e.g. [18]) which means in this case that $\ln(\mathcal{N}(d))$ is concave. It follows that its second derivative is negative and j is monotonically increasing. Additionally we know

$$\lim_{c \rightarrow \infty} j(c) = \delta$$

and

$$\begin{aligned} \lim_{c \rightarrow 0} j(c) &= \lim_{d \rightarrow -\infty} \frac{\alpha - 1}{\|\sigma_L - \theta'\| \sqrt{T}} \frac{\mathcal{N}'(d)}{\mathcal{N}(d)} + \delta \\ &= \lim_{d \rightarrow -\infty} \frac{1 - \alpha}{\|\sigma_L - \theta'\| \sqrt{T}} d + \delta = -\infty. \end{aligned}$$

For the second limit, we use the l'Hôpital rule and the fact that

$$\mathcal{N}''(d) = -d\mathcal{N}'(d).$$

Hence, j changes sign from negative to positive, i.e. w is reverse-S shaped, if $\delta > 0$. $\square$

Appendix D. Replicating strategies for selected pay-offs

We want to calculate replicating strategies for general funding ratios $X^\pm(t, Z_t)$ first. Using Itô's Lemma and the fact that $X^\pm(t, Z_t)$ is a $\mathbb{Q}^L$ -martingale, we know that under the measure $\mathbb{Q}^L$, the SDE has the form

$$dX^\pm(t, Z_t) = Z_t \frac{\partial}{\partial z} X^\pm(t, Z_t) (\sigma_L - \theta') dW_t^{\mathbb{Q}^L}.$$

For the replication, we use the fact that under $\mathbb{Q}^L$ a funding ratio has the form

$$dX^\pm(t, Z_t) = X^\pm(t, Z_t) (\varphi' \sigma - \sigma_L) dW_t^{\mathbb{Q}^L}. \quad (\text{D.1})$$

Thus we have to find a strategy φ that ensures that both SDEs are the same, i.e.

$$X^\pm(t, Z_t)(\varphi(t)'\sigma - \sigma_L) = Z_t \frac{\partial}{\partial z} X^\pm(t, Z_t)(\sigma_L - \theta').$$

If we solve this for φ , we find

$$\begin{aligned} \varphi(t) &= \frac{Z_t}{X^\pm(t, Z_t)} \frac{\partial}{\partial z} X^\pm(t, Z_t)(\sigma')^{-1}(\sigma'_L - \theta) + (\sigma')^{-1}\sigma'_L \\ &= \frac{Z_t}{X^\pm(t, Z_t)} \frac{\partial}{\partial z} X^\pm(t, Z_t)(\varphi^{LH} - \varphi^{PS}) + \varphi^{LH}. \end{aligned} \quad (\text{D.2})$$

This allows us to calculate the replicating portfolios of $(\bar{X}^*)^+$ and $(\bar{X}^*)^-$ resulting from CPT optimization. The corresponding strategies will be denoted by φ_+ and φ_- respectively. To discern the replicating portfolio φ^* of X^* , we need some further calculations as the hedging portfolio of the optimal solution is not the sum of the hedging portfolios of the partial optimal solutions. It rather holds

$$dX^*(t, Z_t) = d(\bar{X}^*(t, Z_t) + B) = d((\bar{X}^*)^+(t, Z_t)) - d((\bar{X}^*)^-(t, Z_t)),$$

since $dB = 0$. Using (D.1), this is the same as

$$\begin{aligned} X^*(t, Z_t)(\varphi'(t)\sigma - \sigma_L)dW_t^{\mathbb{Q}^L} &= ((\bar{X}^*)^+(t, Z_t)(\varphi'_+(t)\sigma - \sigma_L) \\ &\quad - (\bar{X}^*)^-(t, Z_t)(\varphi'_-(t)\sigma - \sigma_L))dW_t^{\mathbb{Q}^L} \end{aligned}$$

or equivalently

$$\begin{aligned} &X^*(t, Z_t)\varphi'(t)\sigma dW_t^{\mathbb{Q}^L} \\ &= [(\bar{X}^*)^+(t, Z_t)\varphi'_+(t)\sigma - (\bar{X}^*)^-(t, Z_t)\varphi'_-(t)\sigma \\ &\quad + (X^*(t, Z_t) - ((\bar{X}^*)^+(t, Z_t) - (\bar{X}^*)^-(t, Z_t)))\sigma_L]dW_t^{\mathbb{Q}^L}. \end{aligned}$$

Note that for all $t \in [0, T]$, it holds

$$X^*(t, Z_t) - ((\bar{X}^*)^+(t, Z_t) - (\bar{X}^*)^-(t, Z_t)) = B.$$

Then the replicating portfolio for X^* is given by

$$\varphi^*(t) = \frac{1}{X^*(t, Z_t)} ((\bar{X}^*)^+(t, Z_t)\varphi_+(t) - (\bar{X}^*)^-(t, Z_t)\varphi_-(t) + B\varphi^{LH}).$$

We use (D.2) for φ_+ and φ_- to state

$$\varphi^*(t) = \frac{Z_t}{X^*(t, Z_t)} \left(\frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t) - \frac{\partial}{\partial z} (\bar{X}^*)^-(t, Z_t) \right) (\varphi^{LH} - \varphi^{PS}) + \varphi^{LH}.$$

In the well-funded case we have $(\bar{X}^*)^- = 0$. Thus the above formula simplifies to

$$\varphi^*(t) = \frac{Z_t \frac{\partial}{\partial z} (\bar{X}^*)^+(t, Z_t)}{X^*(t, Z_t)} (\varphi^{LH} - \varphi^{PS}) + \varphi^{LH}.$$

If the funding ratio is of the form

$$X(T, Z_T) = Z_T^\nu \mathbf{1}_{Z_T \in (c_1, c_2)},$$

with $\nu \in \mathbb{R}$, $c_1 \geq 0, c_2 > 0$, we can derive the funding ratio at time t and the replicating strategy explicitly. We have $Z_T = Z_t Z_{T-t}$. The funding ratio can also be interpreted as the wealth process, discounted with the liability process L . Thus, the funding ratio is a $\mathbb{Q}^L$ -martingale and we have

$$\begin{aligned} X(t, Z_t) &= \mathbb{E}_{\mathbb{Q}^L} [Z_T^\nu \mathbf{1}_{Z_T \in (c_1, c_2)} | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}^L} [Z_{T-t}^\nu Z_t^\nu \mathbf{1}_{Z_t Z_{T-t} \in (c_1, c_2)} | \mathcal{F}_t] \\ &= Z_t^\nu \mathbb{E} [Z_{T-t}^{\nu+1} \mathbf{1}_{Z_{T-t} \in (\frac{c_1}{Z_t}, \frac{c_2}{Z_t})} | \mathcal{F}_t]. \end{aligned}$$

Here we can use the results from Footnote 1 and get

$$\begin{aligned} X(t, Z_t) &= Z_t^\nu \exp \left(-\frac{1}{2}(\nu+1)\|\sigma'_L - \theta\|^2(T-t) + \frac{1}{2}(\nu+1)^2\|\sigma'_L - \theta\|^2(T-t) \right) \cdot \\ &\quad \left(\mathcal{N} \left(\frac{\ln \left(\frac{c_2}{Z_t} \right) + \frac{1}{2}\|\sigma'_L - \theta\|^2(T-t) - (\nu+1)\|\sigma'_L - \theta\|^2(T-t)}{\|\sigma'_L - \theta\|\sqrt{T-t}} \right) \right. \\ &\quad \left. - \mathcal{N} \left(\frac{\ln \left(\frac{c_1}{Z_t} \right) + \frac{1}{2}\|\sigma'_L - \theta\|^2(T-t) - (\nu+1)\|\sigma'_L - \theta\|^2(T-t)}{\|\sigma'_L - \theta\|\sqrt{T-t}} \right) \right) \\ &= Z_t^\nu \exp \left(\frac{1}{2}(\nu+1)\nu\|\sigma'_L - \theta\|^2(T-t) \right) \\ &\quad \left(\mathcal{N} \left(\frac{\ln \left(\frac{c_2}{Z_t} \right) - (\nu + \frac{1}{2})\|\sigma'_L - \theta\|^2(T-t)}{\|\sigma'_L - \theta\|\sqrt{T-t}} \right) \right. \\ &\quad \left. - \mathcal{N} \left(\frac{\ln \left(\frac{c_1}{Z_t} \right) - (\nu + \frac{1}{2})\|\sigma'_L - \theta\|^2(T-t)}{\|\sigma'_L - \theta\|\sqrt{T-t}} \right) \right) \\ &= Z_t^\nu \exp \left(\frac{1}{2}(\nu+1)\nu\|\sigma'_L - \theta\|^2(T-t) \right) \cdot \\ &\quad \left(\mathcal{N} \left(d \left(\frac{c_2}{Z_t}, T-t, \nu \right) \right) - \mathcal{N} \left(d \left(\frac{c_1}{Z_t}, T-t, \nu \right) \right) \right), \end{aligned}$$

with $d(c, s, \nu)$ as in Appendix C. Thus all that is left to do is calculating the derivative of the fair price and putting our results together. The derivative is

$$\begin{aligned} \frac{\partial}{\partial z} X(t, z) &= \exp \left(\frac{1}{2} (\nu + 1) \nu \|\sigma'_L - \theta\|^2 (T - t) \right) \cdot \\ &\quad \left(\nu z^{\nu-1} \left(\mathcal{N} \left(d \left(\frac{c_2}{z}, T - t, \nu \right) \right) - \mathcal{N} \left(d \left(\frac{c_1}{z}, T - t, \nu \right) \right) \right) \right. \\ &\quad \left. - \left(\mathcal{N}' \left(d \left(\frac{c_2}{z}, T - t, \nu \right) \right) - \mathcal{N}' \left(d \left(\frac{c_1}{z}, T - t, \nu \right) \right) \right) \frac{z^{\nu-1}}{\|\sigma'_L - \theta\| \sqrt{T-t}} \right) \\ &= \frac{\nu}{z} X(t, z) - \exp \left(\frac{1}{2} (\nu + 1) \nu \|\sigma'_L - \theta\|^2 (T - t) \right) \cdot \\ &\quad \left(\mathcal{N}' \left(d \left(\frac{c_2}{z}, T - t, \nu \right) \right) - \mathcal{N}' \left(d \left(\frac{c_1}{z}, T - t, \nu \right) \right) \right) \frac{z^{\nu-1}}{\|\sigma'_L - \theta\| \sqrt{T-t}}. \end{aligned}$$

Inserting this and the formula for $X(t, Z_t)$ in (D.2), we see that

$$\begin{aligned} \varphi(t) &= \varphi^{LH} + \left(\nu - \frac{\exp \left(\frac{1}{2} (\nu + 1) \nu \|\sigma_L - \theta'\|^2 (T - t) \right)}{X(t, Z_t) \|\sigma_L - \theta'\| \sqrt{T-t}} \right) \cdot \\ &\quad \left(\mathcal{N}' \left(d \left(\frac{c_2}{Z_t}, T - t, \nu \right) \right) - \mathcal{N}' \left(d \left(\frac{c_1}{Z_t}, T - t, \nu \right) \right) \right) Z_t^\nu (\varphi^{LH} - \varphi^{PS}) \\ &= \varphi^{LH} \\ &\quad + \left(\nu - \frac{1}{\|\sigma_L - \theta'\| \sqrt{T-t}} \frac{\mathcal{N}' \left(d \left(\frac{c_2}{Z_t}, T - t, \nu \right) \right) - \mathcal{N}' \left(d \left(\frac{c_1}{Z_t}, T - t, \nu \right) \right)}{\mathcal{N} \left(d \left(\frac{c_2}{Z_t}, T - t, \nu \right) \right) - \mathcal{N} \left(d \left(\frac{c_1}{Z_t}, T - t, \nu \right) \right)} \right) \cdot \\ &\quad (\varphi^{LH} - \varphi^{PS}) \\ &= \varphi^{LH} \\ &\quad + \left(-\nu + \frac{1}{\|\sigma_L - \theta'\| \sqrt{T-t}} \frac{\mathcal{N}' \left(d \left(\frac{c_2}{Z_t}, T - t, \nu \right) \right) - \mathcal{N}' \left(d \left(\frac{c_1}{Z_t}, T - t, \nu \right) \right)}{\mathcal{N} \left(d \left(\frac{c_2}{Z_t}, T - t, \nu \right) \right) - \mathcal{N} \left(d \left(\frac{c_1}{Z_t}, T - t, \nu \right) \right)} \right) \cdot \\ &\quad (\varphi^{PS} - \varphi^{LH}). \end{aligned}$$

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