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Chapter 8

Pathwise Construction of Affine Processes

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Based on the theory of multivariate time changes for Markov processes, we show how to identify affine processes as solutions of certain time change equations. The result is a strong version of a theorem presented by J. Kallsen in [1] which provides a representation in law of an affine process as a time-change transformation of a family of independent Lévy processes. This also leads to a new perspective on strong approximations for affine processes.

Keywords: affine processes, Lamperti transform, time-change.
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1. Introduction

During the last decades, many alternatives to the Black-Scholes model have been proposed in the literature to overcome its deficiencies. Possible extensions include jumps, stochastic volatility and/or other high dimensional models. Among the most popular ones, we recall the exponential Lévy models, which generalize the Black-Scholes model by introducing jumps. They allow to generate implied volatility smiles and skews similar to the ones observed in the markets. However, in some occasions, independence of increments is too big a restriction. Stochastic volatility models give a way to overcome this problem: when we model the variance parameter in the Black-Scholes model by a CIR model, we get the Heston model, see [2]. The Heston model can be extended by adding jumps in the return component, as in the Bates model (see [3]), and also in the stochastic variance

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component, as in the Barndorff–Nielsen and Shephard model (see [4]). The class of affine processes includes all the above mentioned examples.

Affine processes are a class of time homogeneous Markov processes $X = (X(t))_{t \geq 0}$ taking values in a state space $D \subset \mathbb{R}^d$ characterized by the fact that, for all $(t, x) \in \mathbb{R}_{\geq 0} \times D$, their characteristic function has the following exponential affine form

$$\mathbb{E}^x \left[e^{\langle u, X(t) \rangle} \right] = e^{\varphi(t, u) + \langle x, \psi(t, u) \rangle}, \quad u \in i\mathbb{R}^d,$$

where φ and ψ are two functions taking values in $\mathbb{C}$ and $\mathbb{C}^d$, respectively. The theory of affine processes is dominated by weak characterizations, since affine processes are characterized by a property of their marginal distributions. The functions φ and ψ in the specification of the affine property, solve a system of ODEs, also known in the literature with the name of *generalized Riccati equations*. These equations arise from the regularity property of affine processes. More precisely, in [5] it has been proved that, even on a general state space, stochastically continuous processes having the aforementioned affine property admit a version with càdlàg trajectories. The path regularity implies that the process is a semimartingale with differentiable characteristics up to its lifetime. From this characterization it is possible to conclude differentiability with respect to time of the Fourier–Laplace transform. This property, also called regularity property, is crucial to relate the marginal laws of affine processes with a solution of a system generalized Riccati equations.

This paper is devoted to a pathwise construction of affine processes, when the state space is specified by $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$. The representation proposed in this paper is a multivariate generalization of the Lamperti transformation of Lévy processes in $\mathbb{R}$ with no negative jump. When $D = \mathbb{R}_{\geq 0}$, it has been proved that there exists a one-to-one correspondence between affine processes taking values in D and Lévy processes, see [6]. More precisely, let $Z^{(1)} = (Z^{(1)}(t))_{t \geq 0}$ be a Lévy process starting from 0 taking values in $\mathbb{R}$, whose Lévy measure has support $\mathbb{R}_{\geq 0}$ and let $Z^{(0)}$ be an independent subordinator. In [6], Theorem 2 shows that there exists a solution of the following time–change equation

$$X(t) = x + Z^{(0)}(t) + Z^{(1)} \left(\int_0^t X(s) ds \right)$$

for all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Moreover, it is proved that the solution is a time homogeneous Markov process, taking values in $\mathbb{R}_{\geq 0}$ starting from x , characterized by the property that the logarithm of the characteristic

function of the transition semigroup is given by an affine function of the initial state x . Hence, by definition, it is an affine process taking values in $\mathbb{R}_{\geq 0}$.

In this paper we aim to obtain the analogous result in the multivariate case. In [1] it has been proved that — in distribution — affine processes can be represented by means of $d + 1$ independent Lévy processes taking values in $\mathbb{R}^d$. Under some assumptions on the Lévy triplets, the time change equation

$$X(t) = x + Z^{(0)}(t) + \sum_{i=1}^d Z^{(i)} \left(\int_0^t X_i(s) ds \right), \quad t \geq 0, \quad (1)$$

admits a *weak solution*. Existence of a solution for (1) (also in distribution) holds only under some conditions on the Lévy triplet.

In [1] the following problem is left unsolved: is X a strong solution of the time change equation (1)? In this paper we address this problem using an approximation of the underlying Lévy processes. Recently, an alternative time change construction of affine processes has been presented in [7]. The exposition is organized as follows. In Chapter 2 we provide an overview of some basic results for affine processes. Chapter 3 contains the core of the proof of existence of a strong solution of (1). We show how to pathwisely construct affine processes on the positive orthant solving a family of deterministic ODEs. Then, using the results from Chapter 3, we will see how to construct a solution X of (1) which lives on the same probability space where the Lévy processes are defined. In Chapter 4 we show that, starting from a family of Lévy processes $\{Z^{(k)}\}_{k=0,1,\dots,d}$ specified by some restrictions on their Lévy triplets, the solution time-change equation (1) is a time homogeneous Markov process having the affine property. Observe that, this new existence proof of affine processes gives, as straightforward consequence, the càdlàg property for affine processes.

2. Preliminaries

2.1. Notation

Henceforth D denotes the subset $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ of $\mathbb{R}^d$. The canonical basis of $\mathbb{R}^d$ is denoted by $\{e_i\}_{i=1,\dots,d}$. Given $\Delta \notin D$ define $D_\Delta = D \cup \{\Delta\}$. The set $\mathcal{B}(D)$ is the space of measurable function on D , while $B(D)$ is the space of measurable bounded function on D .

In order to simplify the notation, we introduce the sets of indices I and J defined as

$$I = \{1, \dots, m\} \quad \text{and} \quad J = \{m + 1, \dots, d\}.$$

Moreover, given a set $H \subseteq \{1, \dots, d\}$, the map π_H is the projection of $\mathbb{R}^d$ on the lower dimensional subspace with indices in H .

Comment. In particular

$$\begin{aligned}\pi_I : \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n &\rightarrow \mathbb{R}_{\geq 0}^m \\ x &\mapsto \pi_I x := (x_i)_{i \in I}\end{aligned}$$

and

$$\begin{aligned}\pi_J : \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto \pi_J x := (x_j)_{j \in J}.\end{aligned}$$

Due to the geometry of the state space, the function

$$f_u(x) := e^{\langle x, u \rangle}, \quad x \in D \quad (2)$$

is bounded if and only if u is an element in

$$\mathcal{U} := \mathbb{C}_{\leq 0}^m \times i\mathbb{R}^n, \quad (3)$$

where $\mathbb{C}_{\leq 0}^m = \{w \in \mathbb{C}^m \text{ such that } \operatorname{Re}(w) \in \mathbb{R}_{\leq 0}^m\}$. The notation $\langle \cdot, \cdot \rangle$ with input variables in $\mathbb{R}^d$ denotes the usual scalar product. The same notation is used also when the scalar product is considered in the space $\mathbb{R}^d + i\mathbb{R}^d$. In this case we mean the extension of $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^d + i\mathbb{R}^d$ without conjugation. Unless differently specified, the notation $\mathbb{E}^x[\cdot]$ indicates that the expectation is taken under the probability measure $\mathbb{P}^x$.

Fix $N \in \mathbb{N}$ and let $s \in \mathbb{R}_{\geq 0}^N$. Whenever we are going to consider s as a time parameter, we emphasize its multidimensionality by writing $\underline{s}$. When $\underline{s} = (s_1, \dots, s_N)$ is a multivariate time parameter and X is a stochastic process in $\mathbb{R}^N$, we use the notation

$$\underline{X}(\underline{s}) := (X_1(s_1), \dots, X_N(s_N)) \in \mathbb{R}^N.$$

2.2. Affine processes

In line with the literature, we introduce the affine processes as a class of time homogeneous Markov processes characterized by two additional properties. The first one being stochastic continuity, the second one a condition which characterizes the Fourier–Laplace transform of the one time marginal distributions. This introduction of affine processes is taken from [8], [5] and [9].

Definition 2.1. Let

$$(\Omega, (X(t))_{t \geq 0}, (\mathcal{F}(t))_{t \geq 0}, (p(t))_{t \geq 0}, (\mathbb{P}^x)_{x \in D})$$

be a time homogeneous Markov process. In particular we assume that

- Ω is a probability space,
- $(X(t))_{t \geq 0}$ is a stochastic process taking values in D_Δ ,
- $\mathcal{F}(t) = \sigma(\{X(s), s \leq t\})$ is the filtration generated by canonical coordinates,
- $(p(t))_{t \geq 0}$ is a semigroup of transition functions on $(D_\Delta, \mathcal{B}(D_\Delta))$,
- $(\mathbb{P}^x)_{x \in D_\Delta}$ is a probability measures on $(\Omega, \mathcal{F})$, with $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}(t)$,

satisfying

$$\mathbb{E}^x \left[f(X(t+s)) | \mathcal{F}(t) \right] = \mathbb{E}^{X(t)} \left[f(X(s)) \right], \quad \mathbb{P}^x\text{-a.s. for all } f \in B(D_\Delta). \quad (4)$$

The process X is said to be an *affine process* if it satisfies the following properties:

- for every $t \geq 0$ and $x \in D$, $\lim_{s \rightarrow t} p(s)(x, \cdot) = p(t)(x, \cdot)$ weakly,
- there exist functions $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C}$ and $\psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C}^d$ such that

$$\mathbb{E}^x \left[e^{\langle u, X(t) \rangle} \right] = \int_D e^{\langle u, \xi \rangle} p(t)(x, d\xi) = e^{\varphi(t, u) + \langle x, \psi(t, u) \rangle}, \quad (5)$$

for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$.

Regularity is a key feature for an affine process. It gives differentiability of the Fourier–Laplace transform with respect to time.

Definition 2.2. An affine process X is called *regular* if, for every $u \in \mathcal{U}$, the derivatives

$$F(u) := \partial_t \varphi(t, u) \Big|_{t=0}, \quad R(u) := \partial_t \psi(t, u) \Big|_{t=0}, \quad (6)$$

exist for all $u \in \mathcal{U}$ and are continuous in

$$\mathcal{U}_m = \left\{ u \in \mathbb{C}^d \mid \sup_{x \in D} \operatorname{Re}(\langle u, x \rangle) \leq m \right\},$$

for all $m \geq 1$.

Regularity has been proved in [5]. The proof is based on the fact that affine processes always admit a version which has càdlàg paths. From this path regularity it is possible to conclude differentiability of the Fourier–Laplace transform. We summarize here the main results.

Theorem 2.1 (Theorem 6.4 in [5]). *Every affine process is regular. On the set $\mathbb{R}_{\geq 0} \times \mathcal{U}$, the functions φ and ψ satisfy the following system of generalized Riccati equations:*

$$\begin{aligned}\partial_t \varphi(t, u) &= F(\psi(t, u)), \quad \varphi(0, u) = 0, \\ \partial_t \psi(t, u) &= R(\psi(t, u)), \quad \psi(0, u) = u,\end{aligned}\tag{7}$$

with

$$\begin{aligned}F(u) &= \langle b, u \rangle + \frac{1}{2} \langle u, au \rangle - c \\ &\quad + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) m(d\xi),\end{aligned}\tag{8}$$

$$\begin{aligned}R_k(u) &= \langle \beta_k, u \rangle + \frac{1}{2} \langle u, \alpha_k u \rangle - \gamma_k \\ &\quad + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \langle \pi_{J \cup \{k\}} u, \pi_{J \cup \{k\}} h(\xi) \rangle \right) M_k(d\xi),\end{aligned}\tag{9}$$

for $k = 1, \dots, d$ where here we take as truncation function $h(x) = x \mathbb{1}_{\{|x| \leq 1\}}$. The set of parameters

$$(b, \beta, a, \alpha, c, \gamma, m, M)\tag{10}$$

is specified by

- $b, \beta_i \in \mathbb{R}^d$ for $i = 1, \dots, d$,
- $a, \alpha_i \in S_+^d$ for $i = 1, \dots, d$, where S_+^d denotes the cones of positive semidefinite $d \times d$ matrices,
- $c, \gamma_i \in \mathbb{R}_{\geq 0}$ for $i = 1, \dots, d$,
- m, M_i for $i = 1, \dots, d$ are Lévy measures.

This set of parameters is called *admissible* if the previous conditions are satisfied. The set of admissible parameters fully characterizes an affine process in D .

Remark 2.1. If, additionally, the semigroup of transition functions $(p(t))_{t \geq 0}$ is homogeneous in the space variable, meaning that, for all $x \in D$ and $B \in \mathcal{B}(D)$

$$p(t)(x, B) = p(t)(0, B - x),$$

then necessarily $R = 0$ and it holds

$$\mathbb{E}^x \left[e^{\langle u, X(t) \rangle} \right] = \int e^{\langle u, \xi \rangle} p(t)(x, d\xi) = e^{tF(u) + \langle x, u \rangle},$$

for all $(t, x) \in \mathbb{R}_{\geq 0} \times D$ and $u \in \mathcal{U}$. Hence X is a (possibly killed) Lévy process with Lévy exponent F starting from x .

2.3. Towards the multivariate Lamperti transform

When $D = \mathbb{R}_{\geq 0}$, it has been proved that there exists a one-to-one correspondence between affine processes taking values in D and Lévy processes, see [6].

Comment. More precisely, let $Z^{(1)} = (Z^{(1)}(t))_{t \geq 0}$ be a Lévy process starting from 0 taking values in $\mathbb{R}$ whose Lévy measure has support $\mathbb{R}_{\geq 0}$. This implies that there exists a function $R : i\mathbb{R} \rightarrow \mathbb{C}$ such that

$$\mathbb{E}^0 \left[e^{uZ^{(1)}(s)} \right] = e^{sR(u)},$$

for all $(s, u) \in \mathbb{R}_{\geq 0} \times i\mathbb{R}$. Due to the restrictions on the jump measure, the function R takes the form

$$R(u) = \beta u + \frac{1}{2} \alpha^2 u^2 - \gamma + \int_{\mathbb{R}_{\geq 0}} (e^{u\xi} - 1 - u\xi \mathbb{1}_{\{|\xi| \leq 1\}}) M(d\xi),$$

where $u \in i\mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and M is a measure on $\mathbb{R}_{\geq 0}$ which satisfies

$$\int (1 \wedge |\xi|^2) M(d\xi) < \infty.$$

Moreover, let $Z^{(0)}$ be an independent subordinator with

$$\mathbb{E}^0 \left[e^{uZ^{(0)}(s)} \right] = e^{sF(u)},$$

for all $(s, u) \in \mathbb{R}_{\geq 0} \times i\mathbb{R}$. Since $Z^{(0)}$ is a subordinator, there exists a constant $b \in \mathbb{R}_{\geq 0}$ and a measure m in $\mathbb{R}_{\geq 0}$ satisfying

$$\int (1 \wedge |\xi|) m(d\xi) < \infty,$$

such that, for all $u \in i\mathbb{R}$,

$$F(u) = bu + \int_{\mathbb{R}_{\geq 0}} (e^{u\xi} - 1) m(d\xi).$$

Theorem 2 in [6] shows that there exists a solution of the following time-change equation

$$X(t) = x + Z^{(0)}(t) + Z^{(1)} \left(\int_0^t X(s) ds \right)$$

for all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

The theorem therein shows that the solution of the one-dimensional time change equation is a time homogeneous Markov process, taking values in $\mathbb{R}_{\geq 0}$ starting from x , such that the logarithm of the characteristic function of the transition semigroup is given by an affine function of the initial state x . Hence, by definition, it is an affine process taking values in $\mathbb{R}_{\geq 0}$. Here we are interested in the multivariate generalization of this result, whose weak version is already known in the literature:

Theorem 2.2 (Theorem 3.4 in [1]). *Let X be an affine process with admissible parameter satisfying*

$$\int_{\{|\xi| \geq 1\}} |\xi_k| M_i(d\xi) < \infty \quad \text{and} \quad c = 0, \gamma_i = 0, \quad \text{for } 1 \leq i, k \leq m.$$

On a possibly enlarged probability space, there exist $d+1$ independent Lévy processes $Z^{(k)}$ such that

$$X(t) \stackrel{d}{=} x + Z^{(0)}(t) + \sum_{k=1}^d Z^{(k)} \left(\int_0^t X_k(s) ds \right) \quad t \geq 0. \quad (11)$$

This result has to be understood in distributional sense, because, without any additional assumptions, it is not clear how to conclude that the process X is adapted with respect to the (properly time-changed) filtration generated by the Lévy processes.

In this paper we provide a strong solution of (11) defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ which carries $Z^{(0)}, \dots, Z^{(d)}$.

Remark 2.2. Note that Theorem 2.2 makes use of the theory of martingale problems related to the generator of affine processes to derive a time change representation in law. In this paper, the proof is based on a pathwise approximation and solution, respectively, of the time change equation. Hence, measurability of the solution with respect to the filtration generated by the Lévy process is obtained by construction.

2.4. Affine processes of Heston type

Comment. In this section, we are going to specify a particular subclass of affine processes, which we will call *affine processes of Heston type*. They are characterized by more restrictive admissible parameters but, at the same time, they constitute a canonical family, in the sense that every affine process can be obtained as a pathwise transformation of a canonical one. Instead of stating directly the conditions, we work through an example,

where we point the main motivations and reasonings for the forthcoming Assumptions 2.1–2.3.

Let X be an affine process taking values in D and denote by $(b, \beta, a, \alpha, c, \gamma, m, M)$ its set of admissible parameters. The next condition implies that the function φ in the definition of affine property is identically zero.

Assumption 2.1. *The condition $\mathbf{A}^\omega$ is satisfied if $(b, a, c, m) = (0, 0, 0, 0)$.*

The next assumption implies that the process is homogeneous in the last n variables.

Assumption 2.2. *The condition $\mathbf{A}^H$ is satisfied if, for all $i, j \in J$ it holds $(\beta_i)_j = 0$.*

Finally, to ensure that a solution of the system exists for all $t \geq 0$, we introduce this last set of conditions.

Assumption 2.3. *The condition $\mathbf{\check{A}}$ is satisfied if, for all $i \in I$ it holds $c = 0, \gamma_i = 0$ and*

$$\int (|\pi_I \xi| \wedge |\pi_I \xi|^2) M_i(d\xi), \quad \text{for all } i \in I.$$

Definition 2.3. We call an affine process with admissible parameters $(b, \beta, a, \alpha, c, \gamma, m, M)$ satisfying $\mathbf{A}^\omega$, $\mathbf{A}^H$ and $\mathbf{\check{A}}$ an *affine process of Heston type*.

Among the previous assumptions, only Assumption 2.3 is a real restriction on the structure on the admissible parameters. As observed also in [1] (also compare with [8], Lemma 9.2) Assumption 2.3 guarantees that the solution process does not explode in finite time and hence the time-change process is always well defined. Up to an enlargement of the state space and a pathwise transformation, there is no loss of generality in assuming that both Assumption 2.1 and Assumption 2.2 hold.

In the following proposition we present all steps which allow us to reduce a general affine process into an affine process of Heston type.

3. Existence of the solution of the time-change equation

3.1. The setting

Let $Z^{(1)}, \dots, Z^{(d)}$ be d independent càdlàg $\mathbb{R}^d$ -valued Lévy processes, each of them with Lévy triplet (β_k, α_k, M_k) , $k = 1, \dots, d$, defined on the same probability space.

Now we consider the process $Z = (Z^{(1)}, \dots, Z^{(d)}) \in \mathbb{R}^{d^2}$ on the product space $(\Omega, \mathcal{G}, P)$.

We fix $x \in D$ and consider the functions

$$f_k(y) := x_k + \sum_{i=1}^d y_k^{(i)} \quad \text{for } k = 1, \dots, d, \quad y \in \mathbb{R}^{d^2}. \quad (12)$$

In the next section it will be essential to construct the solution of a system of time-change equations of type

$$Y_i^{(k)}(t) := Z_i^{(k)} \left(\int_0^t f_k(Y(s)) ds \right), \quad k, i = 1, \dots, d, \text{ and } t \geq 0, \quad (13)$$

with $Y = (Y_1^{(1)}, \dots, Y_d^{(1)}, \dots, Y_1^{(d)}, \dots, Y_d^{(d)})$.

Introduce

$$\tau_k(t) := \int_0^t f_k(Y(s)) ds, \quad \text{for } k = 1 \dots, d \text{ and } t \geq 0, \quad (14)$$

and define

$$\underline{\tau}(t) := (\tau_1(t), \dots, \tau_d(t)). \quad (15)$$

Existence of a solution of (13) is equivalent to the existence of a solution of the following system of ODEs

$$\begin{cases} \dot{\tau}_k(t) = f_k(\underline{Z}(\underline{\tau}(t))), & \text{for all } k = 1, \dots, d, t \geq 0, \\ \tau_k(0) = 0, \end{cases} \quad (16)$$

where

$$\underline{Z}(\underline{\tau}(t)) := \left(Z^{(1)}(\tau_1(t)), \dots, Z^{(k)}(\tau_k(t)), \dots, Z^{(d)}(\tau_d(t)) \right).$$

3.2. The core of the proof

We start studying the time-change representation of affine processes in $\mathbb{R}_{\geq 0}^m$. Under the assumption that $n = 0$, (11) reads

$$X(t) \stackrel{d}{=} x + \sum_{i=1}^m Z^{(i)} \left(\int_0^t X_i(s) ds \right) \quad t \geq 0, \quad (17)$$

where $Z^{(1)}, \dots, Z^{(m)}$ are m independent Lévy processes on $\mathbb{R}^m$, each of them with Lévy triplets (β_i, α_i, M_i) , $i = 1 \dots, m$, satisfying

$$\begin{aligned} (\alpha_i)_{kl} &= 0 \text{ for all } k, l \in I \text{ such that } (k, l) \neq (i, i), \\ (\beta_i)_k &\geq 0 \text{ for all } i \in I \text{ and } k \in I \setminus \{i\}. \end{aligned}$$

Suppose that we can show existence of the solution of the following problem

$$\begin{cases} \dot{\tau}_k(t) = x_k + Z_k(\tau(t)), & k = 1, \dots, m, t \geq 0, \\ \tau_k(0) = 0, \end{cases} \quad (18)$$

where

$$Z : \underline{s} \mapsto \sum_{i=1}^m Z^{(i)}(s_i). \quad (19)$$

Then, we can construct X as

$$X_t = x + \sum_{i=1}^m Z^{(i)}(\tau_i), t \geq 0.$$

Observe that (18) in vector notation reads

$$\begin{cases} \dot{\tau}(t) = x + Z(\tau(t)), t \geq 0, \\ \tau(0) = 0. \end{cases} \quad (20)$$

3.2.1. Approximation of the vector field

In order to construct a solution for (20), we seek for a decomposition of type

$$Z = \tilde{Z} + \tilde{\tilde{Z}}$$

such that the system

$$\begin{cases} \dot{\tau}(t) = (x + \tilde{Z})(\tau(t)), t \geq 0 \\ \tau(0) = 0, \end{cases} \quad (21)$$

reduces to a decoupled system of m one dimensional problems and $\tilde{\tilde{Z}} := Z - \tilde{Z}$.

The Lévy–Itô decomposition, together with the canonical form of the admissible parameters, gives

$$\begin{aligned} Z^{(i)}(t) = & \beta_i t + \sigma_i B^{(i)}(t) + \int_0^t \int \xi \mathbb{1}_{\{|\xi| > 1\}} \mathcal{J}^{(i)}(d\xi, ds) \\ & + \int_0^t \int \xi \mathbb{1}_{\{|\xi| \leq 1\}} (\mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi)ds) \end{aligned}$$

where $\sigma_i = \sqrt{(\alpha_i)_{ii}}$, $B^{(i)}$ is a process in $\mathbb{R}^m$ which evolves only along the i -th coordinate as Brownian motion and $\mathcal{J}^{(i)}$ is the jump measure of the process $Z^{(i)}$.

Now, from the assumption on the set of admissible parameter,

$$\pi_{I \setminus \{i\}} \beta_i \in \mathbb{R}_{\geq 0}^{m-1} \quad \text{and} \quad \pi_i \beta_i \in \mathbb{R}.$$

Decompose

$$Z^{(i)} =: \tilde{Z}^{(i)} + \tilde{\tilde{Z}}^{(i)}$$

where $\tilde{Z}^{(i)}$ and $\tilde{\tilde{Z}}^{(i)}$ are two stochastic processes on $\mathbb{R}^m$ defined by

$$\begin{aligned} \tilde{Z}_k^{(i)}(t) &:= 0, \quad \text{for } k \neq i, \\ \tilde{Z}_i^{(i)}(t) &:= \sigma_i B_i^{(i)}(t) + \pi_i \beta_i t + \int_0^t \int \xi_i \mathbb{1}_{|\xi| > 1} \mathcal{J}^{(i)}(d\xi, ds), \\ &\quad + \int_0^t \int \xi_i \mathbb{1}_{|\xi| \leq 1} (\mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi)ds), \\ \tilde{\tilde{Z}}^{(i)}(t) &:= \tilde{\beta}_i t + \int_0^t \int (\xi - \xi_i e_i) \mathbb{1}_{\{|\xi| > 1\}} \mathcal{J}^{(i)}(d\xi, ds), \\ &\quad + \int_0^t \int (\xi - \xi_i e_i) \mathbb{1}_{\{|\xi| \leq 1\}} (\mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi)ds), \end{aligned}$$

where

$$\tilde{\beta}_i = \beta_i - e_i \pi_i \beta_i.$$

Remark 3.1. It is worth noticing that the proposed decomposition stems from the Lévy–Itô decomposition but there is a different separation among the infinite variation jump component and the finite variation part. Indeed, due to the restriction on the admissible parameters, for every $i = 1, \dots, m$ the Lévy measure M_i generates jumps of finite variation on all directions but the i -th. We decompose each Lévy process $Z^{(i)}$ as a spectrally positive Lévy process evolving along the i -th direction (we call it $\tilde{Z}_i^{(i)}$) and a correlated Lévy process of finite variation (the one we call $\tilde{\tilde{Z}}^{(i)}$).

The following lemma, which is an obvious consequence of the restrictions on the admissible parameters, collects some path properties of the processes $\tilde{Z}^{(i)}$ and $\tilde{\tilde{Z}}^{(i)}$. We would like to remark that both càdlàg property and this special structure of the paths are essential ingredients of our proof.

Lemma 3.1. *For all $i = 1, \dots, m$ it holds*

- (1) $\tilde{Z}^{(i)}$ is a Lévy process with no negative jumps,
- (2) $\tilde{\tilde{Z}}^{(i)}$ is a process with increasing paths.

Introduce, for all $\underline{s} \in \mathbb{R}_{\geq 0}^m$,

$$\begin{aligned} \tilde{Z}(\underline{s}) &:= \sum_{i=1}^m \tilde{Z}^{(i)}(s_i), \\ \tilde{\tilde{Z}}(\underline{s}) &:= \sum_{i=1}^m \tilde{\tilde{Z}}^{(i)}(s_i). \end{aligned}$$

We will consider separately the initial value problems with vector fields

$$x + \tilde{\mathcal{Z}} \quad \text{and} \quad \tilde{\mathcal{Z}}.$$

The next result shows that it is possible to find a unique solution for the initial value problem

$$\begin{cases} \dot{\mathcal{I}}((t_0, \tau_0, x); t) = (x + \tilde{\mathcal{Z}})(\mathcal{I}((t_0, \tau_0, x); t)), \\ \mathcal{I}((t_0, \tau_0, x); t_0) = \tau_0. \end{cases}$$

Later, we will show how to construct a solution of the general problem.

Proposition 3.1. *There exists a unique solution of*

$$\begin{cases} \dot{\mathcal{I}}((t_0, \tau_0, x); t) = (x + \tilde{\mathcal{Z}})(\mathcal{I}((t_0, \tau_0, x); t)), \\ \mathcal{I}((t_0, \tau_0, x); t_0) = \tau_0, \end{cases} \quad (22)$$

with $\tau_0 \in \mathbb{R}_{\geq 0}^m$ and $t \geq 0$.

Proof. Observe that (22) is a decoupled system of m equations of type

$$\begin{cases} \dot{\tau}_i((t_0, \tau_0, x); t) = (x_i + \tilde{Z}_i^{(i)})(\tau_i((t_0, \tau_0, x); t)), \quad i = 1, \dots, m, \\ \tau_i((t_0, \tau_0, x); t_0) = \pi_{\{i\}} \tau_0. \end{cases} \quad (23)$$

where each $\tilde{Z}_i^{(i)}$ is a Lévy process with no negative jumps. The existence of a unique solution of (23) follows from [10], Section 6.1. ■

For the proof of the general result, we will need to approximate $\tilde{\mathcal{Z}}$ with piecewise constant functions. Fix $M \in \mathbb{N}$ and consider the partition

$$\mathcal{T}_M := \left\{ \frac{k}{2^M}, \quad k \geq 0 \right\}.$$

Define the following approximations on the partition $\mathcal{T}_M$:

$$\begin{aligned} \uparrow \tilde{\mathcal{Z}}^{(i, M)}(t) &:= \sum_{k=0}^{\infty} \tilde{Z}_{k/2^M}^{(i)} \mathbb{1}_{[\frac{k}{2^M}, \frac{k+1}{2^M})}(t), \\ \downarrow \tilde{\mathcal{Z}}^{(i, M)}(t) &:= \sum_{k=0}^{\infty} \tilde{Z}_{(k+1)/2^M}^{(i)} \mathbb{1}_{[\frac{k}{2^M}, \frac{k+1}{2^M})}(t). \end{aligned}$$

Introduce, for $\underline{s} \in \mathbb{R}_{\geq 0}^m$, the processes $\uparrow \tilde{\mathcal{Z}}^{(M)}(\underline{s})$ and $\downarrow \tilde{\mathcal{Z}}^{(M)}(\underline{s})$ obtained by taking the sums of $\uparrow \tilde{\mathcal{Z}}_{s_i}^{(i, M)}$ and $\downarrow \tilde{\mathcal{Z}}_{s_i}^{(i, M)}$ respectively.

Notation 3.1. Let

$$\Sigma := \bigcup_{i=1}^m \{s \geq 0 \mid \Delta \tilde{Z}^{(i)}(s) > 0\}$$

and augment the partition $\mathcal{T}$ with Σ_M . Denote the family obtained in this way by $\mathcal{T}_M^\Sigma$.

We will first construct a solution for the equation (20) when $\tilde{Z}$ is replaced by $\uparrow \tilde{Z}^{(M)}$.

Hereafter, given $x, y \in \mathbb{R}^m$, we write $x \leq y$ if $x_i \leq y_i$, for all $i = 1, \dots, m$.

3.2.2. The algorithm

Let $\tilde{Z}$ and $\uparrow \tilde{Z}^{(M)}$ be defined as above.

Input: Start by defining the random variables

$$\overleftarrow{\sigma}(\omega) := (0, \dots, 0), \quad (24)$$

$$\overrightarrow{\sigma}(\omega) := (\sigma_1^{(1,M)}(\omega), \dots, \sigma_1^{(m,M)}(\omega)), \quad (25)$$

where each $\sigma_1^{(i,M)}(\omega)$ is the first jump in the path $t \mapsto \uparrow \tilde{Z}^{(i,M)}(t)(\omega)$.

Step 1: Let $\underline{\tau}((t_0, \tau_0, x); t)$ be the solution of the system (22) starting from

$$t_0 = 0, \quad \tau_0 = (0, \dots, 0) \quad \text{and} \quad x \in \mathbb{R}_{\geq 0}^m.$$

Consider the solution of (22) for all times t such that

$$\underline{\tau}((t_0, \tau_0, x); t) < \overrightarrow{\sigma}. \quad (\dagger)$$

Let t^* be the first time such that $(\dagger)$ does not hold anymore. Stop the solution $\underline{\tau}((t_0, \tau_0, x); \cdot)$ at time t^* . Observe that the condition $(\dagger)$ is violated if there exists an index $i^* \in \{1, \dots, m\}$ such that

$$\tau_{i^*}((t_0, \tau_0, x); t^*) = \sigma_1^{(i^*, M)}.$$

Notice here that there might be more than one i^* , where the above equality is valid, however, for the sake of convenience, we assume that there exists only one index for the moment. We will deal with the general case in the proof of Theorem 3.1.

Step 2: Update

$$\overleftarrow{\sigma} := (0, \dots, \sigma_1^{(i^*, M)}, \dots, 0), \quad (26)$$

$$\overrightarrow{\sigma} := (\sigma_1^{(1, M)}, \dots, \sigma_2^{(i^*, M)}, \dots, \sigma_1^{(m, M)}), \quad (27)$$

$$x := x + \Delta \uparrow \tilde{\mathcal{Z}}^{(M)}(\overleftarrow{\sigma}), \quad (28)$$

where $\sigma_2^{(i^*, M)}(\omega)$ is the second jump in the path $t \mapsto \uparrow \tilde{\mathcal{Z}}^{(i^*, M)}(t)(\omega)$.

Step 3: Let $\underline{\tau}((t_1, \tau_1, x_1); t)$ be the solution of the system (22) starting from the *updated* values

$$t_1 = t^*, \quad \tau_1 = \underline{\tau}((t_0, \tau_0, x_0); t^*) \quad \text{and} \quad x_1 = x \in \mathbb{R}_{\geq 0}^m.$$

As before, we let $\underline{\tau}((t_1, \tau_1, x_1); \cdot)$ evolve until

$$\underline{\tau}((t_1, \tau_1, x_1); t^*) < \overrightarrow{\sigma} \quad (29)$$

holds. As soon as this condition does not hold anymore, we stop again the solution.

End: Do iteratively **Step 2** and **Step 3**.

The above algorithm describes the guiding principle for the proof of the next result:

Theorem 3.1. *There exists a solution of*

$$\begin{cases} \dot{\underline{\tau}}^{(M)}((0, 0, x); t) = (x + \tilde{\mathcal{Z}} + \uparrow \tilde{\mathcal{Z}}^{(M)})(\underline{\tau}^{(M)}((0, 0, x); t)), \\ \underline{\tau}^{(M)}((0, 0, x); 0) = 0. \end{cases} \quad (30)$$

Proof. We already did all the main steps for the proof of this result. Let $\mathcal{T}_M$ and Σ be the sets defined in Notation 3.1. Recall that $\mathcal{T}_M^\Sigma$ is a countable family. Enumerate the elements in $\mathcal{T}_M^\Sigma$ such that $\sigma_k^{(i)}$ denotes the k -th jump of $\uparrow \tilde{\mathcal{Z}}^{(i, M)}$. Fix $x \in D$ and set

$$(t_0, \tau_0, x) := (0, 0, x)$$

and

$$\overleftarrow{\sigma} := (0, \dots, 0),$$

$$\overrightarrow{\sigma} := (\sigma_1^{(1, M)}, \dots, \sigma_1^{(i, M)}, \dots, \sigma_1^{(m, M)}),$$

where $\sigma_k^{(i, M)}$ denotes the k -th jump in the path $t \mapsto \uparrow \tilde{\mathcal{Z}}^{(i, M)}(t)$ for all $i = 1, \dots, m$. By definition $\uparrow \tilde{\mathcal{Z}}^{(M)}(\underline{s}) = 0$ for all $\underline{s} < \overrightarrow{\sigma}$. Proposition 3.1

gives the existence of the solution of (22) with this set of input parameters. Denote it by $\underline{\tau}((t_0, \tau_0, x); t)$. As soon as the solution $\underline{\tau}((t_0, \tau_0, x); t)$ reaches a jump time for $\uparrow \tilde{Z}^{(M)}$, the vector field in the equation (30) changes. Precisely, denote by

$$t_1 := \sup\{t > 0 \mid \underline{\tau}((t_0, \tau_0, x); t) < \overrightarrow{\sigma}\}.$$

Again there might be one or more indices i^* , where the condition fails. Collect them in a set $I^* \subseteq \{1, \dots, m\}$. Update the values

$$\begin{aligned} \pi_{I^*} \overleftarrow{\sigma} &:= \pi_{I^*} \overrightarrow{\sigma}, \\ \pi_{I^*} \overrightarrow{\sigma} &:= \pi_{I^*} \overrightarrow{\sigma}_{++}, \end{aligned} \tag{31}$$

where $\overrightarrow{\sigma}_{++}$ contains the next jumps of $\uparrow \tilde{Z}^{(i, M)}$ for all $i \in I^*$ after $\overrightarrow{\sigma}_i$. Then define

$$\begin{aligned} \tau_1 &:= \underline{\tau}((t_0, \tau_0, x); t_1), \\ x_1 &:= x + \Delta \uparrow \tilde{Z}^{(M)}(\overleftarrow{\sigma}). \end{aligned}$$

Now, consider again the solution of (22), but this time with parameters (t_1, τ_1, x_1) . Denote it by $\underline{\tau}((t_1, \tau_1, x_1); t)$ and observe that it is well defined until all the coordinates of $\tau((t_1, \tau_1, x_1); t)$ stay below the next jump times of $\uparrow \tilde{Z}^{(M)}$. We obtain the solution of (22) by pasting a finite amount of solutions obtained in the time subintervals defined by $\mathcal{T}_M^\Sigma$. Define iteratively, for all $n \geq 1$,

$$t_{n+1} := \sup\{t > t_n \mid \underline{\tau}((t_n, \tau_n, x_n); t_n) < \overrightarrow{\sigma}\}, \tag{32}$$

$$\tau_{n+1} := \underline{\tau}((t_n, \tau_n, x_n); t_{n+1}), \tag{33}$$

$$x_{n+1} := x_n + \Delta \uparrow \tilde{Z}^{(M)}(\overleftarrow{\sigma}), \tag{34}$$

where, at each step, $\overleftarrow{\sigma}$ and $\overrightarrow{\sigma}$ are updated using the prescription in (31). Continuity follows by construction. ■

Now that we have found a solution for the approximated problems, we would like to show convergence to the solution of (20).

The following results focus on monotonicity and convergence of (30).

Lemma 3.2. *Let $i = 1, \dots, m$ and $M \in \mathbb{N}$ be fixed. Then, for all $t \geq 0$ it holds*

$$\uparrow \tilde{Z}^{(i, M)}(t) \leq \tilde{Z}^{(i)}(t) \leq \downarrow \tilde{Z}^{(i, M)}(t) \text{ almost surely.}$$

Moreover, for each $\omega \in \Omega$, the sequences $\{\uparrow \tilde{Z}^{(i,M)}(\omega)\}_{M \in \mathbb{N}}$ and $\{\downarrow \tilde{Z}^{(i,M)}(\omega)\}_{M \in \mathbb{N}}$ are monotone in the sense that, for all $t \geq 0$,

$$\uparrow \tilde{Z}^{(i,M+1)}(t)(\omega) \geq \uparrow \tilde{Z}^{(i,M)}(t)(\omega)$$

and

$$\downarrow \tilde{Z}^{(i,M+1)}(t)(\omega) \leq \downarrow \tilde{Z}^{(i,M)}(t)(\omega).$$

Proof. By Lemma 3.1 it holds

$$\tilde{Z}^{(i)}(t) \geq \tilde{Z}_{k/2^M}^{(i)} = \uparrow \tilde{Z}^{(i,M)}(t), \quad \text{a.s. for all } t \in \left[\frac{k}{2^M}, \frac{k+1}{2^M} \right).$$

For the same reason,

$$\tilde{Z}^{(i)}(t) \leq \tilde{Z}_{(k+1)/2^M}^{(i)} = \downarrow \tilde{Z}^{(i,M)}(t), \quad \text{a.s. for all } t \in \left[\frac{k}{2^M}, \frac{k+1}{2^M} \right).$$

Now, since for every $M \in \mathbb{N}$ the partition $\mathcal{T}_{M+1}$ is obtained by halving all the subintervals in the partition $\mathcal{T}_M$, it clearly holds

$$\uparrow \tilde{Z}^{(i,M+1)}(t)(\omega) = \begin{cases} \uparrow \tilde{Z}^{(i,M)}(t)(\omega), & \text{for all } t \in \left[\frac{2k}{2^{M+1}}, \frac{2k+1}{2^{M+1}} \right), \\ \tilde{Z}_{(2k+1)/2^{M+1}}^{(i)}(\omega), & \text{for all } t \in \left[\frac{2k+1}{2^{M+1}}, \frac{2(k+1)}{2^{M+1}} \right). \end{cases}$$

Using again the increasing property of the paths of $\tilde{Z}^{(i)}$ we conclude that

$$\uparrow \tilde{Z}^{(i,M)}(t) \geq \tilde{Z}^{(i)}(t), \quad \text{a.s.}$$

because

$$\tilde{Z}_{(2k+1)/2^{M+1}}^{(i)} \geq \tilde{Z}_{2k/2^{M+1}}^{(i)} = \tilde{Z}_{k/2^M}^{(i)}.$$

The case with $\downarrow \tilde{Z}^{(i,M)}$ goes analogously. ■

Proposition 3.2. Let $M \in \mathbb{N}$ be fixed and denote by $\underline{\tau}^{(M)}((0, 0, x); t)$ the solution of (30) constructed in Theorem 3.1. Then, for all $t \geq 0$ and $x \in \mathbb{R}_{\geq 0}^m$ it holds

$$\underline{\tau}^{(M)}((0, 0, x); t) \leq \underline{\tau}^{(M+1)}((0, 0, x); t), \quad \text{almost surely.}$$

Proof. This follows by construction using the monotonicity proved in Lemma 3.2. Indeed, denote by $\mathcal{T}_M^\Sigma := \{\sigma_k^{(M)}\}_{k \in \mathbb{N}}$ and $\mathcal{T}_{M+1}^\Sigma := \{\sigma_k^{(M+1)}\}_{k \in \mathbb{N}}$ the set of jump times for $\uparrow \tilde{Z}^{(M)}$ and $\uparrow \tilde{Z}^{(M+1)}$ respectively. By construction $\mathcal{T}_M^\Sigma \subset \mathcal{T}_{M+1}^\Sigma$ in the sense that, for each $\sigma_k^{(M)} \in \mathcal{T}_M^\Sigma$ there exists

$h \in \mathbb{N}$ such that $\sigma_k^{(M)} = \sigma_h^{(M+1)} \in \mathcal{T}_{M+1}^\Sigma$. Denote by $\{\sigma_{k_h}^{(M+1)}\}_{h \in \mathbb{N}}$ the jump times of $\uparrow \tilde{Z}^{(M+1)}$ occurring on the subinterval $[\sigma_k^{(M)}, \sigma_{k+1}^{(M)}]$. By construction, there is only one jump inside this interval. Write $\{\sigma_{k_h}^{(M+1)}\}_{h=1, \dots, 3}$ with $\sigma_{k_1}^{(M+1)} = \sigma_k^{(M)}$ and $\sigma_{k_3}^{(M+1)} = \sigma_{k+1}^{(M)}$. Then $\underline{\tau}^{(M+1)}$ is obtained by pasting a finite number of solutions of initial value problems with piecewise linear vector field. For each $h = 1, 2, 3$, $\uparrow \tilde{Z}^{(M+1)}(\sigma_{k_h}^{(M+1)}) \geq \uparrow \tilde{Z}^{(M)}(\sigma_k^{(M)})$. Therefore, on each subinterval $[\sigma_k^{(M)}, \sigma_{k+1}^{(M)}]$, the solution $\underline{\tau}^{(M+1)}((t_k, \tau_k, x_k); t)$ is constructed by pasting a finite number of solutions of type $\underline{\tau}((t_{k,h}, \tau_{k,h}, x_{k,h}); t)$ where $x_{k,h}$ is increasing sequence in h . Hence we conclude that, for all $k \in \mathbb{N}$ and for $t \in [\sigma_k^{(M)}, \sigma_{k+1}^{(M)}]$ it holds

$$\underline{\tau}^{(M+1)}((t_k, \tau_k, x_k); t) \geq \underline{\tau}^{(M)}((t_k, \tau_k, x_k); \sigma_k^{(M)}).$$

■

The last monotonicity argument follows directly from the definition of the ODEs:

Lemma 3.3. *Let M, t_0, τ_0 be fixed and $x \leq y$. Consider the systems*

$$\begin{cases} \dot{\underline{\tau}}^{(M)}((t_0, \tau_0, x); t) = (x + \tilde{Z} + \uparrow \tilde{Z}^{(M)})(\underline{\tau}^{(M)}((t_0, \tau_0, x); t)), \\ \underline{\tau}^{(M)}((t_0, \tau_0, x); t_0) = \tau_0. \end{cases}$$

$$\begin{cases} \dot{\underline{\tau}}^{(M)}((t_0, \tau_0, y); t) = (y + \tilde{Z} + \uparrow \tilde{Z}^{(M)})(\underline{\tau}^{(M)}((t_0, \tau_0, y); t)), \\ \underline{\tau}^{(M)}((t_0, \tau_0, y); t_0) = \tau_0. \end{cases}$$

Then, for all $t \geq t_0$ it holds

$$\underline{\tau}^{(M)}((t_0, \tau_0, x); t) \leq \underline{\tau}^{(M)}((t_0, \tau_0, y); t), \text{ almost surely.}$$

Finally, due to monotonicity, we know that the sequence $\underline{\tau}^{(M)}$ admits a limit. With the next result we show that the limit is actually finite and, by monotone convergence, it coincides with a solution of (20), which are by law unique.

Proposition 3.3. *For all $t \geq 0$ and $x \in \mathbb{R}_{\geq 0}^m$ the sequence $\underline{\tau}^{(M)}((0, 0, x); t)$ converges*

$$\lim_{M \rightarrow \infty} \underline{\tau}^{(M)}((0, 0, x); t) = \underline{\tau}^{(*)}((0, 0, x); t)$$

and the limit can be identified with a solution of (20) almost surely, whose law, however, is unique.

Proof. Let $\underline{\tau}^{(*)}((0, 0, x); \cdot)$ be the limit of the sequence $\{\underline{\tau}^{(M)}((0, 0, x); \cdot)\}_{M \geq 0}$. Since the sequence is a monotone sequence and the limit is continuous, the convergence is actually uniform by Dini's theorem. Observe that the same holds for the limit of the sequence of solutions of the system (30) when $\uparrow \tilde{\mathcal{Z}}^{(M)}$ is replaced by $\downarrow \tilde{\mathcal{Z}}^{(M)}$. Applying the dominated convergence theorem it follows that $\underline{\tau}^{(*)}((0, 0, x); t)$ coincides with a solution of (20). ■

Remark 3.2. We do believe that solutions of Equation (20) are actually unique but we do not need it here and we therefore do not prove this result. It will follow by methods similar to [11], where weak uniqueness leads to pathwise uniqueness. Notice, however, that our time change setting is different from the classical setting.

4. Pathwise construction of affine processes with time-change

In this section we apply the previous results for the pathwise construction of affine processes in $\mathbb{R}_{\geq 0}^m$. Note that this class is also known in the literature under the name of *multitype continuous-state branching process with immigration*. In particular we are going to show that the process constructed in (17) is adapted with respect to the filtration of the underlying Lévy processes. Then, we will see how to generalize the result for affine processes in $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ with $n > 1$.

We start introducing the notion of multivariate filtration and multivariate stopping time taken from [10].

For all $\underline{s} = (s_1, \dots, s_{m^2}) \in \mathbb{R}_{\geq 0}^{m^2}$, define the σ -algebra

$$\mathcal{G}_{\underline{s}} := \sigma \left(\{Z^{(h)}(t_h), t_h \leq s_h, \text{ for } h = 1, \dots, m^2\} \right), \quad (35)$$

where, with an abuse of notation, we denote with $Z^{(h)}$ the h -th coordinate of the process obtained by indexing Z as follows:

$$Z := (Z_1^{(1)}, \dots, Z_m^{(1)}, \dots, Z_1^{(m)}, \dots, Z_m^{(m)}).$$

Then, we complete the σ -algebra by

$$\mathcal{G}_{\underline{s}} = \bigcap_{n \in \mathbb{N}} \mathcal{G}_{\underline{s}^{(n)}} \vee \sigma(\mathcal{N}), \quad (36)$$

where $\mathcal{N}$ is the collection of sets in $\mathcal{G}$ with P -probability zero and $\underline{s}^{(n)}$ is the sequence defined by $s_k^{(n)} = s_k + 1/n$.

Definition 4.1. A random variable $\underline{\tau} = (\tau_1, \dots, \tau_{m^2}) \in \mathbb{R}_{\geq 0}^{m^2}$ is a $(\mathcal{G}_{\underline{s}})$ -stopping time if

$$\{\underline{\tau} \leq \underline{s}\} := \{\tau_1 \leq s_1, \dots, \tau_{m^2} \leq s_{m^2}\} \in \mathcal{G}_{\underline{s}}, \text{ for all } \underline{s} \in \mathbb{R}_{\geq 0}^{m^2}.$$

If $\underline{\tau}$ is a stopping time,

$$\mathcal{G}_{\underline{\tau}} := \{B \in \mathcal{G} \mid B \cap \{\underline{\tau} \leq \underline{s}\} \in \mathcal{G}_{\underline{s}} \text{ for all } \underline{s} \in \mathbb{R}_{\geq 0}^{m^2}\}.$$

Now that we have introduced the necessary notation, we are ready to prove the following result.

Theorem 4.1. *Let τ be a solution of (20).*

(1) *The time-change equation*

$$X(t) = x + \sum_{i=1}^m Z^{(i)}(\theta_i(t)), \quad \text{with } \theta_i(t) = \int_0^t X_i(r) dr, \quad (37)$$

admits a solution.

(ii) *Define*

$$\underline{\theta}^x(t) := (\underbrace{\theta_1(t), \dots, \theta_1(t)}_{m \text{ times}}, \dots, \underbrace{\theta_m(t), \dots, \theta_m(t)}_{m \text{ times}}) \in \mathbb{R}^{m^2}.$$

The random variable $\underline{\theta}^x(t)$ is a $\mathcal{G}_{\underline{s}}$ stopping time for all $t \geq 0$. Hence the time-change filtration

$$\mathcal{G}_{\underline{\theta}^x(t)} := \{A \mid A \cap \{\underline{\theta}^x(t) \leq \underline{s}\} \in \mathcal{G}_{\underline{s}}, \text{ for all } \underline{s} \in \mathbb{R}_{\geq 0}^{m^2}\},$$

is well defined.

(iii) *Let R be the function defined as in (9). The solution of (37) is an affine process with functional characteristics $(0, R)$ with respect to the time-changed filtration $(\mathcal{G}_{\underline{\theta}^x(t)})_{t \geq 0}$. In particular the solution of (37) is unique in law.*

Proof. Let $Y \in \mathbb{R}^{m^2}$ be the process obtained by casting the solutions of (13) as

$$Y := (Y_1^{(1)}, \dots, Y_m^{(1)}, \dots, Y_1^{(m)}, \dots, Y_m^{(m)}).$$

Then

$$X = x + \sum_{k=1}^m Y^{(k)}$$

is a solution of time-change equation (37). Indeed, if $Z_j^{(k)}$ denotes the j -th coordinate of the k -th Lévy process,

$$Z_j^{(k)} \left(\int_0^t f_j^{(k)}(Y(s)) ds \right) = Z_j^{(k)} \left(\int_0^t x_j + \sum_{k=1}^m Y_j^{(k)}(s) ds \right) = Z_j^{(k)} \left(\int_0^t X_k(s) ds \right)$$

and

$$\begin{aligned} X_j(t) &= x_j + \sum_{k=1}^m Y_j^{(k)}(t) \\ &= x_j + \sum_{k=1}^m Z_j^{(k)} \left(\int_0^t X_k(s) ds \right). \end{aligned}$$

Now we move on the measurability of the time-change process. Observe

$$\underline{\tau} := (\underbrace{\tau_1, \dots, \tau_1}_{m \text{ times}}, \dots, \underbrace{\tau_m, \dots, \tau_m}_{m \text{ times}}) \in \mathbb{R}^{m^2}.$$

where

$$\tau_k(t) = \int_0^t f_k(Y(s)) ds$$

is a $\mathcal{G}_{\underline{s}}$ stopping time for all $t \geq 0$. This follows from [10], Theorem VI.2.2.. From the affine relationship between X and Y we conclude that $\underline{\theta}(t)$ is a $\mathcal{G}_{\underline{s}}$ stopping time and therefore the time-changed filtration is well defined. Now, we need to check that X is a homogeneous Markov process with respect to $(\mathcal{G}_{\underline{\theta}^x(t)})_{t \geq 0}$. Applying [12], Proposition I.6 at each component $Z^{(k)}$, $k = 1, \dots, d$ we get that $(\underline{Z}(\underline{\theta}^x(t+h)) - \underline{Z}(\underline{\theta}^x(t)))_{h \geq 0}$ has the same law as $\underline{Z}(\underline{\theta}^x(h))_{h \geq 0}$ and it is independent of $\mathcal{G}_{\underline{\theta}^x(t)}$.

Therefore

$$\begin{aligned} X_{t+h}^x &= X^x(t) + N(\underline{Z}(\underline{\theta}^x(t+h)) - \underline{Z}(\underline{\theta}^x(t))) \\ &=: \mathcal{S}(t)(\underline{Z}(\underline{\theta}^x(t+h)) - \underline{Z}(\underline{\theta}^x(t)), X^x(t)), \end{aligned}$$

with

$$\begin{aligned} \mathcal{S}(t) : (\mathbb{R}^{m^2}, \prod_{i=1}^m (\mathcal{G}_{\underline{\theta}^x(t)})) \times (\mathbb{R}^m, \mathcal{G}_{\underline{\theta}^x(t)}) &\rightarrow (\mathbb{R}^m, \mathcal{G}_{\underline{\theta}^x(t)}) \\ (Z, X) &\rightarrow X + \sum_{i=1}^m Z^{(i)}. \end{aligned}$$

Therefore, we conclude that the conditional law of X_{t+h}^x , given $\mathcal{G}_{\underline{\theta}^x(t)}$, is $X^x(t)$ measurable. Markov property translates into

$$X_{t+h}^x = \mathcal{S}_0(\underline{Z}(\underline{\theta}_h^y), y)|_{y=X^x(t)}.$$

Additionally the time-change process is absolutely continuous with

$$\frac{d}{dt}\theta_i^{(k)}(t) = X_k(t^-), \text{ for all } k, i = 1, \dots, m.$$

The characteristics of the time-changed semimartingale can be computed using the formulas in [13], Theorem 8.4. Then we conclude that the process $(\mathcal{S}_0(\underline{Z}(\theta^x(t)), x))_{t \geq 0}$ has characteristics $(\beta(X_-), \alpha(X_-), M(X_-))$, where

$$\begin{aligned}\beta(x) &= x_1\beta_1 + \dots + x_m\beta_m, \\ \alpha(x) &= x_1\alpha_1 + \dots + x_m\alpha_m, \\ M(x, B) &= x_1M_1(B) + \dots + x_mM_m(B), \quad B \in \mathcal{B}(D).\end{aligned}$$

■

The result can be extended very easily for processes in $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ under the assumption that the d Lévy processes additionally satisfy

- (A1) for all $k = 1, \dots, m$ $Z_j^{(k)}(t)$ is a deterministic process for all $j = m+1, \dots, d$,
 (A2) for all $k = m+1, \dots, d$ $Z_j^{(k)}$ is constantly equal to zero for all $j = 1, \dots, d$.

Indeed, if $\tau(t) = (\tau_1(t), \dots, \tau_m(t))$ denotes as before the solution of (16) restricted to $\pi_I Z^{(k)}$ for all $k = 1, \dots, m$, then

$$\begin{cases} \dot{\tau}_k(t) = f_k(\underline{Z}(\tau(t))), t \geq 0, \\ \tau_k(0) = 0, \end{cases} \quad (38)$$

admits a solution for all $k = 1, \dots, d$. To show this, it suffices to check that for each $k = 1, \dots, d$, it holds

$$\begin{aligned}\tau_k(t) &= \int_0^t \left(x_k + \sum_{i=1}^d Y_k^{(i)}(s) \right) ds \\ &= \int_0^t \left(x_k + \sum_{i=1}^d Z_k^{(i)}(\tau_i(s)) \right) ds.\end{aligned}$$

In particular, if we have a solution for the system

$$\tau_k(t) = \int_0^t \left(x_k + \sum_{i=1}^m Z_k^{(i)}(\tau_i(s)) \right) ds \quad \text{for all } k = 1, \dots, m,$$

then, for all $t \geq 0$ and $j = m+1, \dots, d$, we can compute

$$\tau_j(t) = \int_0^t \left(x_j + \sum_{i=1}^m Z_j^{(i)}(\tau_i(s)) \right) ds.$$

Observe that the right hand side of the last equation does not depend anymore on the left hand side.

Comment. Before to show how to deal with the most general case, we clarify the main steps by means of an easy two dimensional example with $n = m = 1$.

Example 4.1. Suppose that $m = n = 1$. If we have a solution for the time change equation

$$Y_i^{(k)}(t) := Z_i^{(k)}(\tau_k) = Z_i^{(k)}\left(\int_0^t f_k(Y(s))ds\right), \quad \text{for } k, i = 1, 2 \text{ and } t \geq 0.$$

Inserting the definitions of the $Y_i^{(k)}$, it is clear that X satisfies

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} Z_1^{(1)}(\int_0^\cdot X_1(s)ds) + Z_1^{(2)}(\int_0^\cdot X_2(s)ds) \\ Z_2^{(1)}(\int_0^\cdot X_1(s)ds) + Z_2^{(2)}(\int_0^\cdot X_2(s)ds) \end{pmatrix}.$$

In vector notation, we can write

$$X = x + \sum_{i=1}^2 Z^{(i)}\left(\int_0^\cdot X_i(s)ds\right)$$

which is indeed the formulation in Theorem 2.2.

However, the previous argument shows that the existence of a solution τ can be generalized in a straightforward way in $\mathbb{R}^d$ only under the additional condition (A1) and (A2). Observe that (A1) is not really a restriction because it follows by the conditions on the admissible parameters. The next proposition shows that, up to a pathwise transformation, also (A2) is not a real restriction. Moreover we show that, up to an enlargement of the state space, any affine process with non zero F can be viewed as an affine process with no state-independent component up to an enlargement of the state space.

Proposition 4.1. *Let X be a conservative affine process taking values on $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$. On a possibly enlarged probability space, there exists a process X^ϖ such that*

- (1) X^ϖ is an affine process taking values in $\mathbb{R}_{\geq 0}^{m+1} \times \mathbb{R}^n$ satisfying the following property: there exists a function $\psi^\varpi: \mathbb{C}_{\leq 0}^{m+1} \times i\mathbb{R}^n \rightarrow \mathbb{C}^{d+1}$ such that for all $(t, x^\varpi) \in \mathbb{R}_{\geq 0} \times (\mathbb{R}_{\geq 0}^{m+1} \times \mathbb{R}^n)$ it holds

$$\mathbb{E}^{x^\varpi}\left[e^{\langle u, X^\varpi \rangle}\right] = e^{\langle x^\varpi, \psi^\varpi(t, u) \rangle}$$

for $u \in \mathbb{C}_{\leq 0}^{m+1} \times i\mathbb{R}^n$,

(2) for all $u = (u_1, u_2) \in \mathbb{C}_{\leq 0}^{m+1} \times i\mathbb{R}^n$ it holds

$$\pi_{\{m+2, \dots, d+1\}} \psi^\varpi(t, u_1, u_2) = u_2,$$

for all $t \geq 0$,

(3) the set of admissible parameters for X^ϖ satisfies

- $(b, a, c, m) = (0, 0, 0, 0)$,
- for all $i, j \in J$ it holds $\pi_j(\beta_i) = 0$,
- for all $i \in I$ it holds

$$\int (|\pi_I \xi| \wedge |\pi_I \xi|^2) M_i(d\xi), \quad \text{for all } i \in I,$$

- for all $k = 1, \dots, m+1$, the matrix α_k has the form

$$\alpha_k = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ 0 \dots 0 & (\alpha_k)_{kk} & 0 \dots 0 & 0 & \dots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ \hline 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \alpha_k^J \end{array} \right),$$

with $(\alpha_k)_{kk} \geq 0$ and $\alpha_k^J \in S_n^+$.

(4) for all $(t, x) \in \mathbb{R}_{\geq 0} \times D$ and $u \in \mathcal{U}$, define $x^\varpi = (1, x)$ and $v = (0, u)$. It holds

$$\mathbb{E}^x \left[e^{\langle u, X(t) \rangle} \right] = \mathbb{E}^{x^\varpi} \left[e^{\langle v, X^\varpi(t) \rangle} \right].$$

Proof. Given two indices $i, j = 1, \dots, d+1$ with $i < j$ denote by

$$[i : j] := \{i, i+1, \dots, j-1, j\}.$$

We start using [14], Proposition 1.23. Fix $x_0 \in \mathbb{R}_{\geq 0}$ and define

$$x^\varpi := (x_0, x) \in \mathbb{R}_{\geq 0}^{m+1} \times \mathbb{R}^n, \quad (39)$$

$$\mathcal{U}^\varpi := \mathbb{C}_{\leq 0}^{m+1} \times i\mathbb{R}^n, \quad (40)$$

$$\psi^\varpi(t, u_0, u_1, \dots, u_d) := \begin{pmatrix} \varphi(t, u_1, \dots, u_d) + u_0 \\ \psi(t, u_1, \dots, u_d) \end{pmatrix}. \quad (41)$$

Due to regularity in t of $\varphi(t, u)$ and $\psi(t, u)$, we conclude that $\psi^\infty(t, \cdot)$ is a regular semiflow. Hence, from [8], Proposition 7.4, we conclude that there exists an affine process X^∞ with state space $\mathbb{R}_{\geq 0}^{m+1} \times \mathbb{R}^n$ satisfying

$$\mathbb{E}^{x^\infty} \left[e^{\langle u, X^\infty(t) \rangle} \right] = e^{\langle x^\infty, \psi^\infty(t, u) \rangle}, \quad u \in \mathcal{U}^\infty.$$

Now we can apply the method of moving frames (see [15], Theorem 5.1) to the affine process X^∞ . Let $(0, \beta, 0, \alpha, 0, 0, 0, M)$ be its set of admissible parameters. Denote by $\mathcal{B}$ the $d \times d$ matrix obtained by placing each β_i , $i = 1, \dots, d$ as a column

$$\mathcal{B} = \left(\begin{array}{c|c} \mathcal{B}_I & 0 \\ \hline \mathcal{B}_{IJ} & \mathcal{B}_J \end{array} \right). \quad (42)$$

Define the matrix

$$T = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \mathcal{B}_J^\top \end{array} \right) \in \mathbb{R}^{d \times d}$$

and the map

$$\mathcal{T} : X^\infty \mapsto X^\infty - T^\top \int_0^\cdot X^\infty(s) ds.$$

The process $\mathcal{T}X^\infty$ is an affine process with Fourier–Laplace transform given by

$$\mathbb{E}^{x^\infty} \left[e^{\langle u, \mathcal{T}X^\infty(t) \rangle} \right] = e^{\langle \pi_{[1:m+1]} x^\infty, \pi_{[1:m+1]} \psi^\infty(t, u) \rangle + \langle \pi_{[m+2:d+1]} x^\infty, \pi_{[m+2:d+1]} u \rangle}.$$

In particular, the first three conditions are satisfied. Now we move on the structure of the matrices α_k , $k = 1, \dots, m+1$. Due to the restrictions on the admissible parameters, α_1 is already in the specified form with $(\alpha_1)_{11} = 0$. The matrices α_k , $k = 1, \dots, m+1$ can be transformed simultaneously into block diagonal form by means of a linear map. See [16]. Finally, if $v = (0, u)$ with $u \in \mathcal{U}$

$$\mathbb{E}^{(1,x)} \left[e^{\langle v, X^\infty(t) \rangle} \right] = e^{\varphi(t,u) + \langle x, \psi(t,u) \rangle} = \mathbb{E}^x \left[e^{\langle u, X(t) \rangle} \right],$$

within

$$\psi^\infty(t, (0, u)) := \begin{pmatrix} \varphi(t, u) \\ \psi(t, u) \end{pmatrix}. \quad (43)$$

■

To summarize, only the equation determining $\pi_I X$ is a real time-change equation. As soon as we provide a strong solution for the system of time-change equations describing the positive components, we automatically find a solution for the components taking values in $\mathbb{R}^n$.

We illustrate the previous proposition with an example

Example 4.2. Let us start by writing (11) componentwise. Denote by $Z_j^{(k)}$ the j -th coordinate of the k -Lévy process. Then (11) reads

$$\begin{aligned} X_1(t) &= x_1 + Z_1^{(0)}(t) + \sum_{k=1}^d Z_1^{(k)} \left(\int_0^t X_k(s) ds \right), \quad t \geq 0 \\ &\vdots \\ X_d(t) &= x_d + Z_d^{(0)}(t) + \sum_{k=1}^d Z_d^{(k)} \left(\int_0^t X_k(s) ds \right), \quad t \geq 0. \end{aligned} \quad (44)$$

Due to the drift conditions, we conclude that, for $k = m+1, \dots, d$, $Z^{(k)}$ is a Lévy process with triplet $(\beta_k, 0, 0)$ with $\pi_I \beta_k$ identically zero. Therefore we can write

$$\begin{aligned} X_1(t) &= x_1 + Z_1^{(0)}(t) + \sum_{k=1}^m Z_1^{(k)} \left(\int_0^t X_k(s) ds \right), \quad t \geq 0 \\ &\vdots \\ X_d(t) &= x_d + Z_d^{(0)}(t) + \sum_{k=1}^m Z_d^{(k)} \left(\int_0^t X_k(s) ds \right) \\ &\quad + \sum_{k=m+1}^d \pi_d \beta_k \left(\int_0^t X_k(s) ds \right), \quad t \geq 0. \end{aligned}$$

We first transform the process into another affine process with functional characteristic $F = 0$. Just for simplicity assume that $n = m = 1$. Augment the process $X = (X_1, X_2)$ by considering

$$\overline{X} = (\overline{X}_0, \overline{X}_1, \overline{X}_2) := (1, X_1, X_2).$$

Moreover define, for $k = 0, 1, 2$,

$$(\overline{Z}_k^{(0)}, \overline{Z}_k^{(1)}, \overline{Z}_k^{(2)}) := (0, Z_k^{(1)}, Z_k^{(2)}).$$

Then we can write

$$\begin{pmatrix} \overline{X}_0(t) \\ \overline{X}_1(t) \\ \overline{X}_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} + \overline{Z}^{(0)} \left(\int_0^t \overline{X}_0(s) ds \right) + \overline{Z}^{(1)} \left(\int_0^t \overline{X}_1(s) ds \right) \\ + \overline{Z}^{(2)} \left(\int_0^t \overline{X}_2(s) ds \right), \quad t \geq 0.$$

Observe that the process $\overline{X}$ takes values in $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}$. Hence, up to a change of the state space, we are led to consider solutions of

$$\begin{aligned} X_1(t) &= x_1 + \sum_{k=1}^m Z_1^{(k)} \left(\int_0^t X_k(s) ds \right), \quad t \geq 0 \\ &\vdots \\ X_d(t) &= x_d + \sum_{k=1}^m Z_d^{(k)} \left(\int_0^t X_k(s) ds \right) + \sum_{k=m+1}^d \pi_d \beta_k \left(\int_0^t X_k(s) ds \right), \quad t \geq 0. \end{aligned}$$

The second pathspace transformation additionally simplifies the system and allows us to work only with affine processes with admissible parameters satisfying the additional property $\pi_k \beta_j = 0$ for all $j, k \in J$. This means that the Lévy processes $Z^{(k)}$ with $k = m+1, \dots, d$ are not only deterministic but actually identically equal to zero (as required by the assumption (A2)). This transformation has been introduced in [15] and it is based on the method of the moving frames. Here we focus on the case $n = m = 1$. Let $X = (X_1, X_2)$ be an affine process in $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Consider the process $Y = (Y_1, Y_2)$ with $Y_0 = x$ and

$$\begin{aligned} Y_1(t) &:= X_1(t) - \int_0^t X_1(s) ds, \quad t \geq 0 \\ Y_2(t) &:= X_2(t) - (\beta_2)_2 \int_0^t X_2(s) ds, \quad t \geq 0. \end{aligned}$$

From [15], Theorem 5.1 guarantees that Y is again an affine process in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with admissible parameter $\beta_2^Y = (0, 0)$. Moreover this transformation can be inverted. Hence, when $n = m = 1$, up to an invertible pathspace transformation, we can restrict ourselves to the solution of a system of type

$$\begin{aligned} X_1(t) &= x_1 + Z_1^{(1)} \left(\int_0^t X_1(s) ds \right), \quad t \geq 0 \\ X_2(t) &= x_2 + Z_2^{(1)} \left(\int_0^t X_1(s) ds \right), \quad t \geq 0, \end{aligned}$$

or more generally

$$\begin{aligned} X_1(t) &= x_1 + \sum_{k=1}^m Z_1^{(k)} \left(\int_0^t X_k(s) ds \right), \quad t \geq 0 \\ &\vdots \\ X_d(t) &= x_d + \sum_{k=1}^m Z_d^{(k)} \left(\int_0^t X_k(s) ds \right), \quad t \geq 0. \end{aligned}$$

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