Innovations in Insurance, Risk- and Asset Management

Kathrin Glau
Daniël Linders
Aleksey Min
Matthias Scherer
Lorenz Schneider
Rudi Zagst

editors

KPMG Center of Excellence
in Risk Management

World Scientific

Chapter 4

Examples of Wrong-Way Risk in CVA Induced by Devaluations on Default

Damiano Brigo

*Dept. of Mathematics – Mathematical Finance Group & Stochastic Analysis Group –
Imperial College,
London, SW7 2AZ, U.K.
damiano.brigo@imperial.ac.uk
www.imperial.ac.uk*

Nicola Pede

*Dept. of Mathematics – Mathematical Finance Group & Stochastic Analysis Group –
Imperial College,
London, SW7 2AZ, U.K.
n.pede13@imperial.ac.uk
www.imperial.ac.uk*

When calculating *Credit Valuation Adjustment* (CVA), the interaction between the portfolio's exposure and the counterparty's credit worthiness is referred to as *Wrong-Way Risk* (WWR). Making the assumption that the Brownian motions driving both the market (exposure) and the (counterparty) credit risk-factors dynamics are correlated represents the simplest way of modeling the dependence structure between these two components. For many practical applications, however, such an approach may fail to account for the right amount of WWR, thus resulting in misestimates of the portfolio's CVA. We present a modeling framework where a further — and indeed stronger — source of market/credit dependence is introduced through devaluation jumps on the market risk-factors' dynamics. Such jumps happen upon the counterparty's default and are a particularly realistic feature to include in case of sovereign or systematically important counterparties. Moreover, we show that, in the special case where the focus is on FX/credit WWR, devaluation jumps provide an effective way of incorporating market information coming from quanto Credit Default Swap (CDS) basis spreads and we derive the corresponding CVA pricing equations as a system of coupled PDEs.

Keywords: credit default swaps, liquidity spread, liquidity premium, credit liquidity correlation, liquidity pricing, intensity models, reduced form models, capital asset pricing model, credit crisis, liquidity crisis.

MSC Codes: 60H10, 60J60, 91B70.

1. Introduction

Credit Valuation Adjustment (CVA) is a risk adjustment to the fair value of a portfolio of derivative contracts that reflects the credit risk of the common counterparty with which such contracts have been agreed. It accounts for the potential losses incurred due to the default of the counterparty before the contracts' expiries, and, as such, it heavily depends on the correct modeling of the credit risk factors, the market risk factors, and of the interaction between them. One of the main challenges in calculating CVA is indeed constituted by the lack of liquid market data to be used to infer risk-neutral credit/market joint distributions.

The calibration and approximation techniques showed in this paper can be used, for example, to connect currency devaluation with multi-currency Credit Default Swap (CDS) par-spreads (see [1] for more details) and that, in turn, allows to calculate CVA more accurately. The resulting FX/credit cross modeling improvement is crucial where the interaction between the counterparty credit and the FX is strong, i.e. with emerging market credits and systemically relevant counterparties where the right/wrong wayness is more relevant.

Throughout this work, we will often refer to the interaction between market and credit risk factors as *Wrong-Way Risk* (WWR).

1.1. Overview of the modeling framework

In this work, we will be using *unilateral*, as opposed to bilateral, CVA to illustrate the impact of WWR modeling. 'Unilateral' in this context means that we will be assuming only the counterparty to be a defaultable entity, while we neglect our own default risk. This assumption effectively amounts to considering ourselves as a default-free entity. On the one hand, the assumptions made in a bilateral framework are more realistic (both counterparties are subject to default risk), but, on the other hand, they introduce additional complexity in the form of the default time/default time interaction that, we think, might obfuscate the main points with respect to WWR modeling that this article wants to illustrate. We will use a probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t, t \geq 0))$ satisfying the usual hypotheses. In particular $(\mathcal{F}_t, t \geq 0)$ is a filtration under which the dynamics of the risk factors are adapted and under which the default time of the reference entity is a stopping-time.

In the setting just described, CVA can be represented through the following formula:

$$\text{CVA}_t = \mathbb{E}_t \left[\left(\phi(\tau, X_\tau^{(0)}, \dots, X_\tau^{(N)}) \right)^+ \frac{B_t}{B_\tau} \mathbb{1}_{\{\tau \leq T\}} \right] \quad (1)$$

where

- $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ is the expected value calculated with respect to the filtration $\mathcal{F}_t$;
- $(t, x_0, \dots, x_N) \mapsto \phi(t, x_0, \dots, x_N)$ is the value of the portfolio at time t and for a realization $x_0, \dots, x_N$ of the market risk-factors;
- τ is the counterparty's default time;
- $(B_t, t \geq 0)$ is the numeraire associated to the pricing measure.

For the same reason, and given that the main examples shown throughout this work focus on equity/credit WWR and FX/credit WWR, we will neglect the randomness in interest rates and just assume deterministic interest rate term-structures. We will generically refer to the FX and equity related risk-factors as the *market risk-factors*.

The plan of the work is the following: in Section 2 we will present a PDE approach based on reduced-form framework for credit risk modeling. We will show how to handle the case where the market risk-factor is a jump-diffusion process and how to link the jump times to the default time of the counterparty. In Section 4 we will show how this approach is able to provide a more effective way to model WWR.

For the equity case, in Section 3 we will present an alternative credit modeling framework called AT1P and we will show how it naturally links equity and credit risk factors. In Section 4 we will compare this approach to the jump-diffusion approach in reduced-form framework.

We refer to [2] and [3] for a general overview on CVA modeling with applications to multiple asset classes. In [2], both unilateral and bilateral CVA calculation frameworks are described. Techniques for calculations of extreme CVA values, in the context of bilateral CVA modeling, have been recently shown in [4].

2. A PDE approach for both FX-driven and equity-driven WWR

In this section we present a modeling approach to handle the case where the counterparty's hazard rate is stochastic and where one additional market risk-factor is modeled as a jump-diffusion process whose only jump occurs upon the counterparty's default.

This approach can be effectively applied to both the case where the market risk-factor is an FX rate and to the case where the market risk-factor is an equity asset or index. Test cases with respect to both the examples are presented in Section 4.

2.1. *FX*

Let us consider the existence of multiple risk-neutral pricing measures, each of them linked to a specific currency. In this context, we will denote by $(B(t), t \geq 0)$ the money market account denominated in the (arbitrarily chosen) domestic currency, while we will denote by $(\hat{B}(t), t \geq 0)$ the money market account denominated in another foreign currency. Both of them are assumed deterministic. Furthermore, we will denote by Z_t the spot FX rate expressing the cost of one unit of foreign currency in the domestic currency and we will be using $(D_t, t \geq 0)$ for the default process:

$$D_t = \mathbb{1}_{\{\tau \leq t\}}, \quad t \geq 0. \quad (2)$$

Let us consider the following specification for the dynamics of $(Z_t, t \geq 0)$ and of the counterparty's hazard rate, $(\lambda_t, t \geq 0)$:

$$dY_t = a(b - Y_t) dt + \sigma^Y dW_t^Y, \quad t \geq 0, \quad (3)$$

$$Y_0 = y, \quad (4)$$

$$dZ_t = \mu^Z Z_t dt + \sigma^Z Z_t dW_t^Z + \gamma^Z Z_{t-} dD_t, \quad t \geq 0, \quad (5)$$

$$Z_0 = z, \quad (6)$$

$$d\langle W^Y, W^Z \rangle_t = \rho dt, \quad t \geq 0, \quad (7)$$

$$\lambda_t = e^{Y_t}, \quad t \geq 0. \quad (8)$$

An application of the generalized Itô formula (see [5]) allows us to write the $\mathbb{Q}$ -dynamics of $(CVA_t, t \geq 0)$. Using $f(t, Z_t, Y_t, D_t) = CVA_t$:

$$\begin{aligned} df = & \partial_t f dt + \partial_z f \left(\mu^Z z dt + \sigma^Z z dW_t^Z + \gamma^Z z dD_t \right) \\ & + \partial_y f \left(a(b - Y_t) dt + \sigma^Y dW_t^Y \right) + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f d[Z, Z]_t \\ & + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f d[Y, Y]_t + \rho \sigma^Z \sigma^Y z \partial_{zy} f d[Z, Y]_t \\ & + \Delta f \Delta D_t - \partial_z f \Delta Z_t \end{aligned} \quad (9)$$

where, with some abuse of notation, we have defined the jump-to-default term as

$$\Delta f := f(t, Z_{t-} + \Delta Z_t, Y_t, D_{t-} + \Delta D_t) - f(t, Z_{t-}, Y_t, D_{t-}). \quad (10)$$

Definition of Δf . Δf depends on the jumps of $(Z_t, t \geq 0)$ and $(D_t, t \geq 0)$. The two jump components, however, are driven by a common jump driver $((D_t, t \geq 0)$ itself, see Eq. (5)), and the jumps in the FX rate dynamics are given by

$$\Delta Z_t = \gamma^Z Z_{t-} \Delta D_t. \quad (11)$$

It must be noted that $(D_t, t \geq 0)$ starts at 0 and jumps to 1 at a single time, τ , upon default. This means, in particular, that $D_{t-} + \Delta D_t$ takes a value different from zero only upon default, and that, for all the times previous to that, the following equation holds:

$$D_t = 0, \quad t < \tau. \quad (12)$$

The first term in Eq. (10) can then be rewritten as

$$f(t, Z_{t-} + \Delta Z_t, Y_t, D_{t-} + \Delta D_t) = f(t, Z_{t-} + \Delta Z_t, Y_t, \Delta D_t) \quad (13)$$

and, considering also Eq. (11), the equation for Δf can be written as

$$\Delta f = f(t, (1 + \gamma^Z) Z_{t-}, Y_t, 1) - f(t, Z_{t-}, Y_t, 0). \quad (14)$$

Compensated martingale for $(D_t, t \geq 0)$. A compensator for $(D_t, t \geq 0)$ in the measure $\mathbb{Q}$ is defined as the process $(A_t, t \geq 0)$ such that $D_t - A_t$ is a $\mathbb{Q}$ -martingale with respect to $(\mathcal{F}_t, t \geq 0)$. The compensator for $(D_t, t \geq 0)$ is given by (see Lemma 7.4.1.3 in [5])

$$dA_t = \mathbb{1}_{\{\tau > t\}} \lambda_t dt. \quad (15)$$

We define the resulting martingale as $(M_t, t \geq 0)$, where

$$M_t = D_t - A_t. \quad (16)$$

Consequently, the compensator of the term $\Delta f \Delta D_t$ in Eq. (9) can be written as

$$(1 - D_t) e^{Y_t} \Delta f dt, \quad (17)$$

which, conditional on $\mathcal{F}_t$, $D_t = d$, $Z_{t-} = z$, and $Y_t = y$, is equal to

$$(1 - d) e^y \left(f(t, z(1 + \gamma^Z), y, 1) - f(t, z, y, 0) \right) dt. \quad (18)$$

2.1.1. No-arbitrage drift for the market risk-factor (FX)

Any specification of the FX rate dynamics is subject to arbitrage constraints. One way to formulate them is by requiring that the Radon–Nikodym derivative defined by

$$L_t = \frac{Z_t \hat{B}_t}{Z_0 B_t}, \quad L_0 = 1. \quad (19)$$

is a martingale. The drift specification that satisfies such condition is provided by

$$\mu^Z = r(t) - \hat{r}(t) - \lambda_t \gamma^Z \mathbb{1}_{\{r > t\}} = r(t) - \hat{r}(t) - \lambda_t \gamma^Z (1 - D_t), \quad (20)$$

where $r(t)$ and $\hat{r}(t)$ are the — assumed deterministic — domestic and foreign short rates, respectively.

FX symmetry. Each FX rate links two risk-neutral pricing measures and, in deciding how to set its no-arbitrage drift, we arbitrarily started from one of them. We could as well have started from the other risk-neutral pricing measure. An argument equivalent to the one discussed in the previous paragraph would lead, in this case, to set a drift condition for the process $(X_t, t \geq 0)$ defined as $X_t = \frac{1}{Z_t}$. When the FX rate is specified as a geometric Brownian motion, it does not matter if we start from one measure or from the other one, as the two approaches lead to consistent results.

Despite the introduction of the jump in the FX rate dynamics, the consistency between $(X_t, t \geq 0)$ and $(Z_t, t \geq 0)$ is maintained. This is stated in the next proposition.

Prop 2.1 (FX symmetry and devaluation jumps). *Let us consider an FX rate process whose dynamics in the domestic risk-neutral measure $\mathbb{Q}$ is specified by Eq. (5) and whose drift is given by Eq. (20). Then the dynamics of the process $(X_t, t \geq 0)$ where $X_t = 1/Z_t$ in the foreign risk-neutral measure $\hat{\mathbb{Q}}$ is given by*

$$\begin{aligned} dX_t &= (\hat{r} - r)X_t dt - \sigma^Z X_t d\hat{W}_t^Z + X_{t-} \gamma^X d\hat{M}_t, \\ X_0 &= \frac{1}{z}, \end{aligned} \quad (21)$$

where the devaluation rate for $(X_t, t \geq 0)$ is given by

$$\gamma^X = -\frac{\gamma^Z}{1 + \gamma^Z} \quad (22)$$

and where $(\hat{M}_t, t \geq 0)$ is the martingale defined in Eq. (16) expressed in $\hat{\mathbb{Q}}$. In particular, (21) is such that the Radon–Nikodym derivative $(\hat{L}_t, t \geq 0)$ defined by

$$\hat{L}_t = \frac{B_t}{\hat{B}_t} \frac{X_t}{X_0}, \quad \hat{L}_0 = 1. \quad (23)$$

is a $\hat{\mathbb{Q}}$ -martingale.

A proof of this proposition is presented in [1].

It is now possible to write a Feynman–Kac PDE to compute the value of $CVA_t(T)$. Indeed, $(CVA_t, t \geq 0)$ is a $\mathbb{Q}$ -price and, as such, it must locally grow at the rate r . Therefore, its drift must satisfy the following equation:

$$\begin{aligned} \partial_t f + \left(r - \hat{r} - \lambda_t \gamma^Z (1 - d) \right) z \partial_z f + a(b - Y_t) \partial_y f \\ + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f + \rho \sigma^Z \sigma^Y z \partial_{zy} f \\ + e^y (1 - d) \Delta f = r f \, dt, \end{aligned}$$

where the explicit dependence of f on the state variables (x, y, t, d) has been omitted for clarity of reading. It is worth noting that, if it wasn't for the last term, this would be the typical PDE for default-free payoffs. Incidentally, this jump-to-default term is also the only term of the equation where the two default-specific components $f(t, (1 + \gamma^Z)z, y, 1)$ and $f(t, z, y, 0)$ appear together. In fact, by conditioning first on $d = 1$ and then on $d = 0$ we can decouple the two functions

$$u(t, z, y) := f(t, (1 + \gamma^Z)z, y, 1) \quad (24)$$

$$v(t, z, y) := f(t, z, y, 0) \quad (25)$$

and calculate them by solving iteratively two separate — lower dimension — PDE problems. We first solve for u , as for $d = 1$ the last term does not appear in the equation, and, once u has been calculated, we use it to solve for v .

Remark 2.1 (Interpretation of u and v). *The functions u and v account for the post-default and pre-default values respectively of a derivative with payoff $\phi(x, y, d)$. The price of this derivative can be written as*

$$V_t = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = d], \quad (26)$$

where, due to the strong Markov property of the processes $(X_t, t \geq 0)$, $(Y_t, t \geq 0)$, and $(D_t, t \geq 0)$, the expected value on the right-hand side

can be written as

$$f(t, x, y, d) = \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = d]. \quad (27)$$

This can be decomposed as $f(t, x, y, d) = \mathbb{1}_{\{d=1\}}u(t, x, y) + \mathbb{1}_{\{d=0\}}v(t, x, y)$ where

$$v(t, x, y) := \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 0], \quad (28)$$

$$u(t, x, y) := \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 1], \quad (29)$$

in fact

$$\begin{aligned} f(t, x, y, d) &= \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = d] \\ &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 0] \\ &\quad + \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 1] \\ &= \mathbb{1}_{\{\tau > t\}} v(t, x, y) + \mathbb{1}_{\{\tau \leq t\}} u(t, x, y) \end{aligned} \quad (30)$$

as both $\mathbb{1}_{\{\tau > t\}}$ and $\mathbb{1}_{\{\tau \leq t\}}$ are measurable in the $\mathcal{F}_t$ filtration. The derivative price can then be written as

$$V_t = \mathbb{1}_{\{\tau > t\}} v(t, X_t, Y_t) + \Delta D_t u(t, X_t, Y_t), \quad (31)$$

where we defined

$$\Delta D_t := \mathbb{1}_{\{\tau > t\}} - \mathbb{1}_{\{\tau > t-\}}. \quad (32)$$

2.1.2. Final conditions — CVA payoff

The final conditions for functions u and v depend on the portfolio for which the CVA is going to be calculated. For the sake of explanation, let us consider a stylized portfolio where we expect to receive a single — deterministic and constant in time — cash-flow payment from our counterparty at maturity $T > 0$. The payment will be settled in a different currency from the one used to determine the risk-neutral pricing measure. Furthermore, we consider null interest rates. Under these assumptions, the CVA is given by:

$$\text{CVA}_0 = \mathbb{E}_0 [Z_\tau \mathbb{1}_{\{\tau \leq T\}}]. \quad (33)$$

The final conditions for the two functions can be written as:

$$u(T, z, y) = f(T, (1 + \gamma^Z)z, y, 1) = (1 + \gamma^Z)z, \quad (34)$$

$$v(T, z, y) = f(T, z, y, 0) = 0. \quad (35)$$

Remark 2.2 (Terminal condition for u). *In order to get a better understanding of the conditions set above, it might be useful to write the terminal condition for u in terms of conditioned expected values:*

$$u(T, z, y) = f(T, (1 + \gamma^Z)z, y, 1) \\ = \mathbb{E} \left[Z_\tau \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{F}_T, \tau \leq T \right] = Z_\tau. \quad (36)$$

It is also worth recalling that $u(t, z, y)$ is only needed in solving v in the term representing the jump-on-default component Δf , that is, the change in value given to a default happening at t (see Remark 2.1).

The PDE problem that must be solved to obtain u is then given by

$$\begin{aligned} \partial_t u = & -(r - \hat{r})z\partial_z u - a(b - y)\partial_y u - \frac{1}{2} \left(\sigma^Z x \right)^2 \partial_{zz} u \\ & - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} u - \rho \sigma^Z \sigma^Y z \partial_{zy} u, \end{aligned} \quad (37a)$$

$$u(T, z, y) = (1 + \gamma^Z)z. \quad (37b)$$

Once the solution to this problem has been calculated, it can be used to solve the PDE for v , which is then given by

$$\begin{aligned} \partial_t v = & -(r - \hat{r})z\partial_z v - a(b - y)\partial_y v - \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} v \\ & - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} v - \rho \sigma^Z \sigma^Y z \partial_{zy} v \\ & + e^y (v - u - \gamma^Z z \partial_z v), \end{aligned} \quad (38a)$$

$$v(T, z, y) = 0. \quad (38b)$$

Remark 2.3. From the PDE system (38) above it is clear why it makes sense to define $u(t, z, y) := f(t, (1 + \gamma^Z)z, y, 1)$ rather than $u(t, z, y) := f(t, z, y, 1)$ as we need the term in Δf .

2.2. Equity

We can use the same modeling approach presented in Section 2.1 to calculate the CVA of an equity portfolio. Similarly to the previous case, we consider an exponential Ornstein–Uhlenbeck process for the stochastic hazard rate and a geometric Brownian motion with a deterministic relative jump occurring upon the counterparty’s default for the other relevant risk-factor — in this case, equity, rather than FX.

We specify the model as:

$$dS_t = \mu^S S_t dt + \sigma^S S_t dW_t^S + \gamma^S S_{t-} dD_t, \quad (39)$$

$$S_0 = s_0, \quad (40)$$

$$dY_t = a(b - Y_t) dt + \sigma^Y dW_t^Y, \quad (41)$$

$$Y_0 = y_0, \quad (42)$$

$$d\langle W^S, W^Y \rangle_t = \rho dt, \quad (43)$$

where the stochastic intensity of default $(\lambda_t, t \geq 0)$ is given — as in the FX/credit case — by

$$\lambda_t = e_t^Y. \quad (44)$$

2.2.1. No-arbitrage drift for the market risk-factor (equity)

The no-arbitrage condition on $(S_t, t \geq 0)$'s drift is given by requiring that its discounted price is a martingale. In formulas:

$$\mathbb{E}_t \left[\frac{S_T}{B_T} \right] = \frac{S_t}{B_t}. \quad (45)$$

Under the assumption of deterministic interest rates, that translates into

$$\mu^S = r(t) - \gamma^S(1 - D_t)\lambda_t. \quad (46)$$

2.2.2. Final conditions — CVA payoff

In this case, the prototype portfolio we are interested in studying is made of a single put option and it can be written as

$$V_t = (1 - R)\mathbb{E}_t \left[(K - S_\tau)^+ \mathbb{1}_{\{\tau \leq T\}} \right]. \quad (47)$$

This is a classic example of a position carrying a high level of WWR, and therefore highly sensitive to correlation assumptions. In order to get some intuition about the correlation effect in this type of trade, it might be useful to consider the limit case where the equity option's underlying is the counterparty's stock. In case of, for example, financial problems of the reference entity, these will be reflected in their balance account, and, arguably, in their stock price (that will decrease) and in their credit quality (that will decrease). Both these changes will affect the CVA of the position in the same direction (it will increase) and, moreover, they can reinforce each other. Therefore, taking in account their joint effect can have a dramatic effect on our calculations.

Given the strong Markov property of all the processes involved, the CVA can be assumed to be a function of their values at valuation time:

$$f(t, s, y, d) := (1 - R)\mathbb{E}_t \left[(K - S_\tau)^+ \mathbb{1}_{\{\tau \leq T\}} \right]. \quad (48)$$

The PDE system that must be solved to calculate the value above can be deduced using the same techniques showed in Section 2.1 in the FX/credit case. Let us then define the conditioned on default and on survival values of f

$$u(t, s, y) := f(t, s(1 + \gamma^S), y, 1), \quad (49)$$

$$v(t, s, y) := f(t, s, y, 0), \quad (50)$$

so that the final conditions on the newly defined functions are given by

$$u(T, s, y) := f(T, s, y, 1) = (1 - R)(K - s), \quad (51)$$

$$v(T, s, y) := f(T, s, y, 0) = 0. \quad (52)$$

The same equations can therefore be used, with two important differences to take into account:

- i) the no-arbitrage drift of the market risk-factor (equity in this case, rather than FX) is given by Eq. (46) rather than Eq. (20)
- ii) the terminal conditions are given by Equations (51) and (52).

The resulting PDE system is then given by:

$$\begin{aligned} \partial_t u &= -rs\partial_s u - a(b - y)\partial_y u - \frac{1}{2} \left(\sigma^S x \right)^2 \partial_{ss} u \\ &\quad - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} u - \rho \sigma^S \sigma^Y s \partial_{sy} u \end{aligned} \quad (53a)$$

$$u(T, s, y) = (1 - R)(K - s). \quad (53b)$$

Once the solution to this problem has been calculated, it can be used to solve the PDE for v , which is then given by

$$\begin{aligned} \partial_t v &= -(r)s\partial_s v - a(b - y)\partial_y v - \frac{1}{2} \left(\sigma^S s \right)^2 \partial_{ss} v \\ &\quad - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} v - \rho \sigma^S \sigma^Y s \partial_{sy} v \\ &\quad + e^y (v - u - \gamma^S s \partial_s v) \end{aligned} \quad (54a)$$

$$v(T, s, y) = 0. \quad (54b)$$

3. A structural approach for equity/credit WWR

In contrast to the reduced-form approach presented in Section 2.2, the model that we show in this section is based on a *structural* approach to credit modeling. Structural models seem a particularly good candidate to model WWR in equity as they naturally link credit risk and stock prices through their knowledge of balance account quantities. Furthermore, in the equity case, differently from the FX rate case, we don't have the constraint provided by the existence of quanto CDS basis. On the one hand, this means that we have more freedom on modeling the underlying process dynamics. For the sake of illustration, we will be using a simple *Geometric Brownian Motion* (GBM) to do that. On the other hand, by doing that we lose one "correlation" parameter (the devaluation-on-default parameter γ), thus delegating the whole WWR effect to the instantaneous correlation between the equity process and the credit process. The effects on WWR of this modeling choice are compared with reduced-form based models both in a purely diffusive case and in a jump-diffusion setting in Section 4 showing that, from a WWR perspective, a structural approach lies in between the other two modeling approaches.

3.1. AT1P

We will be using a model based on *Analytically Tractable First-Passage* (AT1P) to model the dependence between equity and credit in this section. AT1P was first presented in [6] where the authors extended the original Merton and Black Cox setting in two important ways:

- i) by considering a deterministic non-flat barrier (see Eq. (57) below);
- ii) by allowing for a time-dependent volatility for the firm value process (see Eq. (56) below).

$$\tau = \inf\{t \geq 0 : V_t \leq H(t)\}, \quad \inf \emptyset := +\infty. \quad (55)$$

This approach has been used by [7] for pricing Lehman equity swaps taking into account counterparty risk. In that work, the use of a random default barrier associated with misreporting and risk of fraud was also considered. Here we use the first version of the model, having deterministic non-flat barriers.

The reason for imposing a particular shape for the barrier and for considering different maturities for the outstanding debt of the firm is to make the calibration and pricing processes as feasible and practical as possible,

as, under these assumptions, the survival probabilities of the firm can be recovered using closed form formulas. Indeed, the main achievement of the AT1P generalization relies in the possibility of calibrating the model to the whole CDS term structure of the firm, as illustrated for Parmalat and Lehman in [6], [7].

In addition to those works, we refer to [8] for the deduction of results on the pricing formulas for one-touch barriers in AT1P and to [9] for an application of AT1P to *Contingent Conversion* (CoCo) bond pricing, where a technique to calibrate AT1P to the spot stock price, the entity Tier-1 Capital Ratio, and the CDS spreads is introduced.

We present here a formulation of the model in the simplified setting where — in addition to the debt barrier being deterministic — the dividends and the risk free short rate are constant. The firm-value process is specified by the following SDE:

$$dV_t = (r - q)V_t dt + \sigma(t)V_t dW_t^V, \quad V_0 = v, \quad (56)$$

while a time-dependent barrier is parameterized as

$$\hat{H}(t) = H e^{(r-q)t-B \int_0^t \sigma^2(s) ds}, \quad \hat{H}(0) = H_0. \quad (57)$$

Next, we look separately at how this model can be used to model both the credit risk and the equity component when calculating CVA for an equity trade.

3.1.1. Credit risk

Survival probabilities are given in closed-form expression as

$$\mathbb{Q}(\tau > T) = \Phi(d_1) - \left(\frac{H}{V_0}\right)^{2B-1} \Phi(d_2), \quad (58)$$

where τ is defined in (55) and where

$$d_1 := \frac{\log \frac{V_0}{H} + \frac{2B-1}{2} \int_0^T \sigma(s)^2 ds}{\left(\int_0^T \sigma(s)^2 ds\right)^{1/2}},$$

$$d_2 := d_1 - \frac{2 \log \frac{V_0}{H}}{\left(\int_0^T \sigma(s)^2 ds\right)^{1/2}}.$$

Closed-form formulas for survival probabilities are sufficient for CDS calibration in a single-currency framework and when credit is assumed to be independent of interest rates.

3.1.2. Equity price

In AT1P an entity can default at any time and not only at its debt maturity. If we assume that the debt still has a clear single final maturity T and that early default is given by safety covenants, it is not unreasonable to model equity as an option on the firm value with maturity T that is killed if the default barrier is reached before T (see also [2], Chapter 8). We calculate the stock price E_t (in this framework) as a down-and-out European call option, that is

$$E_t = B(t)\mathbb{E}_t \left[\frac{\left(V_T - \hat{H}(T) \right)^+ \mathbb{1}_{\{\tau > T\}}}{B(T)} \right] = f(t, V_t). \quad (59)$$

This equation can be used both to calculate the stock price both inside a simulation of an equity-dependent payoff and in the calibration procedure.

A closed form solution for the price of the option is given for example in [8], see also the Equity chapter in [2]. The formula is as follows:

$$\begin{aligned} f = \frac{B(t)}{B(T)} & \left(V_t e^{\int_t^T (v(s) + \frac{\sigma(s)^2}{2}) ds} (1 - \Phi(d_3)) \right. \\ & \quad \left. - \hat{H}(T) (1 - \Phi(d_4)) \right. \\ & \quad \left. - \hat{H}(t) \left(\frac{\hat{H}(t)}{V_t} \right)^{2B} e^{\int_t^T (v(s) + \frac{\sigma(s)^2}{2}) ds} (1 - \Phi(d_5)) \right. \\ & \quad \left. + \hat{H}(T) \left(\frac{\hat{H}(t)}{V_t} \right)^{2B-1} (1 - \Phi(d_6)) \right), \end{aligned} \quad (60)$$

where

$$\begin{aligned} v(t) &= r - q - \frac{\sigma(t)^2}{2}, \\ d_3 &= \frac{\left(\log \frac{\hat{H}(T)}{\hat{H}} \right)^+ - \log \frac{V_t}{\hat{H}} - \int_t^T (v(s) + \sigma(s)^2) ds}{\left(\int_t^T \sigma(s)^2 ds \right)^{1/2}}, \\ d_4 &= \frac{\left(\log \frac{\hat{H}(T)}{\hat{H}} \right)^+ - \log \frac{V_t}{\hat{H}} - \int_t^T v(s) ds}{\left(\int_t^T \sigma(s)^2 ds \right)^{1/2}}, \end{aligned}$$

$$d_5 = \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{\hat{H}(t)^2}{H V_t} - \int_t^T (v(s) + \sigma(s)^2) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}},$$

$$d_6 = \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{\hat{H}(t)^2}{H V_t} - \int_t^T (v(s)) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}.$$

3.2. Introducing WWR

We introduce WWR in AT1P by allowing the option's underlying process, $(S_t, t \geq 0)$, to be correlated to the firm's stock price process. Similarly to the case analysed in Section 2.2 — but with the crucial difference that in this case we don't assume a jump-to-default component — the dynamics of the option's underlying process is specified as a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma^S S_t dW_t^S, \quad S_t = S_0. \quad (61)$$

The instantaneous correlation between processes $(S_t, t \geq 0)$ and $(V_t, t \geq 0)$ is in principle difficult to estimate, because it mixes together information of fundamentally different nature: a stock price S , that is observable and traded, and a firm-value V , that is not traded, and that is only observable at most quarterly or at any planned balance account public disclosure. The link provided by firm-value and stock price in AT1P Eq. (60), however, proves useful in this respect, because it allows, as a first approximation, to use the empirical correlation between the firm's stock price and the underlying equity process as a firm-value/underlying equity's correlation estimate. In formulas, we will be assuming that:

$$d \langle W^S, W^V \rangle = d \langle W^S, W^E \rangle. \quad (62)$$

4. Results

In this section, we show results produced by all the models described in the previous sections and we compare the different “WWR power” that each model is able to provide. From an asset class perspective, we will present tests on FX/credit interaction and on equity/credit interaction. From a modeling perspective, we will test impact on WWR of instantaneous correlation in both reduced-form model and in structural model (the latter only in equity case). We will also test the devaluation jump impact on WWR.

We will use acronyms to refer to models. We will use *exponential Ornstein–Uhlenbeck* (expOU) for the reduced-form models, and we will denote by a (+J) the addition of jumps to default on the market risk factor. We will use AT1P for the structural model. A test/asset class summary is presented in Table 1.

Table 1. Summary of the models tested in Section 4.

Test case	Reduced-form model		Structural model
	expOU	expOU+J	AT1P
FX	✓	✓	
Equity	✓	✓	✓

4.1. *Models calibrations*

For the tests presented in this section we considered a dummy set of market data. Specifically, we used

- a flat term-structure or CDS par-spreads (200 bps);
- a flat term-structure of zero rates (100 bps);
- we fixed the lognormal volatility of both equity and FX rate to 20%;
- in the reduced-form approach, we fixed the the normal volatility of the OU process driving the stochastic hazard rate to 50% and its speed of mean reversion speed to 0.001.

For the AT1P calibration, we refer to [7], [6] for further discussions and examples around calibration. We also refer to [9] for a more recent example and for a detailed description on how to use balance account information to fix the barrier level H_0 (see Eq. (57)).

For the expOU model calibration, we refer in particular to [1] where calibration is discussed extensively and where 3 year daily calibration outputs relying on quanto CDS spreads are presented.

4.2. *Equity WWR: Correlation impact*

In Figure 1 we present the correlation impact on CVA calculation of a portfolio consisting of a single trade: an ATM put option with expiry 1 year.

Fig. 1. Correlation impact in the structural AT1P approach versus the reduced-form expOU approach. The CVA values obtained in case of credit-market independence (highlighted in both plots) are the same.

The agreement between the expOU and AT1P graphs on the zero correlation case provides a good safety check on the two model implementations.

It is worth highlighting that we used a common x-axis for the two charts, as indeed both the CVA calculated using expOU and using AT1P depend on a correlation parameter. The correlation parameter, however, has a very different interpretation in the two cases. As discussed in Section 3.2, in AT1P a possible proxy for it is provided by the equity/equity correlation between the counterparty and the option's underlying. In expOU, instead, the correlation to be used is the one between the counterparty's credit and the option's underlying. Unsurprisingly, the two parameters have opposite impact on CVA. Given the fact that the stochastic factor driving the asset value in AT1P is more directly linked to the counterparty's default time (through Eq. (55)) than the stochastic hazard rate in expOU, AT1P's correlation parameter has a stronger impact than expOU's correlation on WWR.

4.3. Equity WWR: Devaluation impact

The impact of the devaluation factor γ^S on the CVA of a portfolio constituted of a single 1-year expiry at-the-money equity put option is shown in Figure 2. The maximum CVA values that could be produced through correlation both in this reduced-form framework (for negative equity/credit

correlation) and in AT1P (for positive equity/equity correlation) have been highlighted in Figure 2.

Figure 2 illustrates an unsurprising behavior of CVA — and that applies to credit modeling in general: the more a parameter is directly linked to the default time definition, the higher its impact in terms of WWR. This phenomenon is akin to tranche pricing, where correlation on stochastic hazard rate turns up being a much less effective mechanism to price CDO tranche than a copula approach (see, for example, [10]).

Fig. 2. Devaluation jump impact in reduced-form (BK) approach.

4.4. *FX WWR: FX Vega*

In this section, we show a less intuitive, yet of high practical importance, effect that correlation has on CVA pricing. For this example, we considered a portfolio constituted of a single cash payment that is settled in a currency different from the one in which the numeraire is denominated (like in Eq. (33)). We used a reduced-form approach to calculate the CVA with no devaluation jumps on the FX rate. The correlation impact on FX vega is plotted on Figure 3, showing that, when FX and credit are independent, the CVA has no FX Vega. This is not the case when instead we introduce correlation between FX and credit. The practical impact of having or not having correlation/dependence is in this case quite relevant as it would drastically change the hedging strategy to be used to hedge the CVA risk.

Fig. 3. Correlation impact on FX vega in a reduced-form model (BK).

We can explain the results above in formulas: when FX and credit are independent — and considering zero interest rates for simplicity — the CVA from Eq. (33) can be written as:

$$\begin{aligned} \text{CVA}_0 &= \mathbb{E}_0 \left[X_\tau \mathbb{1}_{\{\tau \leq T\}} \right] = \int_0^T \mathbb{E}_0 [X_s] \mathbb{Q}(s \leq \tau < s + ds) \\ &= X_0 \mathbb{Q}(\tau \leq T) \end{aligned} \quad (63)$$

showing in fact no dependence on $(X_t, t \geq 0)$ dynamics. It is worth nothing that the same would happen also for more complicated derivatives as far as their only dependence on the FX rate is given by a mismatch in the pricing/payment currencies. Let us consider a derivative paying off an amount $\phi(Y_T)$ at T if the counterparty has not defaulted before T , where $(Y_t, t \geq 0)$ is a non-FX market risk factor. CVA can be written in this case as:

$$\begin{aligned} \text{CVA}_0 &= \mathbb{E}_0 \left[\phi(Y_T) X_\tau \mathbb{1}_{\{\tau \leq T\}} \right] \\ &= \int_0^T \mathbb{E}_0 [\phi(Y_s) X_s] \mathbb{Q}(s \leq \tau < s + ds) \\ &= X_0 \int_0^T \hat{\mathbb{E}}_0 [\phi(Y_s)] \mathbb{Q}(s \leq \tau < s + ds) \end{aligned} \quad (64)$$

where we denoted as $\hat{\mathbb{E}}_0[\cdot]$ the expected value calculated in the payment currency pricing measure. The formula shows, again, no dependence on $(X_t, t \geq 0)$ dynamics.

5. Conclusions

In the present work, we investigated different modeling approaches to account for the credit/market dependence in CVA pricing. We focused on two distinct cases, one where the only market risk-factor was the FX rate and one where the only market risk-factor was the equity asset. In both cases, we showed how to calculate CVA in a PDE framework using a reduced-form approach for credit modeling and we investigated the impact that the inclusion of a default-driven devaluation jump to the market risk-factor dynamics has on WWR. Furthermore, in the equity case, we showed how a structural approach can also be effectively used to model the WWR.

Consistently with our intuition, the more directly a given modeling approach links the credit risk component to the market risk component, the higher is the variation in WWR that can be achieved by adopting it. In particular, delegating the whole credit/market dependence to instantaneous correlation provides the smallest range of WWR variation, while the inclusion of a jump-to-default effect on the market risk factor provides the largest one, with the structural approach — in the equity case — lying somewhere in between these two extreme cases.

In future works, we plan to include testing with more realistic portfolios, where impacts coming from variations on moneyness can be studied and, possibly, we will introduce a modeling approach that could consistently account for the observed skew in the market risk factor.

References

1. D. Brigo, N. Pedersen and A. Petrelli, Multi-Currency Credit Default Swaps: Quanto Effects and FX Devaluation Jumps https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2703605, (2015).
2. D. Brigo, M. Morini and A. Pallavicini, *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for all Asset Classes* (Wiley, 2013).
3. G. Cesari and L. Gordon, *Modelling, Pricing, and Hedging Counterparty Credit Exposure* (Springer, 2006).
4. M. Scherer and T. Schulz, Extremal dependence for bilateral credit valuation adjustments, *International Journal of Theoretical and Applied Finance* **19**, 7 (2016).
5. M. Jeanblanc, M. Yor and M. Chesney, *Mathematical Methods for Financial Markets* (Springer, 2009).

6. D. Brigo and M. Tarengi, Credit Defaultly Swap Calibration and Equity Swap Valuation under Counterparty Risk with a Tractable Structural Model, in *Proceedings of the FEA 2004 Conference at MIT*, 2004.
7. D. Brigo, M. Morini and M. Tarengi, Credit Calibration with Structural Models and Equity Return Swap valuation under Counterparty Risk, in *Credit Risk Frontiers: Sub- prime crisis, Pricing and Hedging, CVA, MBS, Ratings and Liquidity*, ed. Bielecki, T., Brigo, D., and Patras, F. (Wiley, 2010) pp. 457–484.
8. F. Rapisarda, Barriers Option on Underlyings with Time-Dependent Parameters: a Perturbation Approach https://papers.ssrn.com/sol3/papers.cfm?abstract_id=682184, (2003), Technical report, available at SSRN.com.
9. D. Brigo, J. Garcia and N. Pede, CoCo Bond Pricing with Credit and Equity Calibrated First-Passage Firm Value Models, *International Journal of Theoretical and Applied Finance* **18**, 3 (2015) pp. 1550015–1550030.
10. D. Brigo and F. Mercurio, *Interest Rate Models — Theory and Practice: With Smile, Inflation and Credit*, second edn. (Springer, 2006).