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Kathrin Glau
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Rudi Zagst

editors

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Chapter 2

Wrong-Way Risk Adjusted Exposure: Analytical Approximations for Options in Default Intensity Models

Damiano Brigo

*Department of Mathematics, Imperial College London,
London SW7 2AZ, UK
damiano.brigo@imperial.ac.uk*

Thomas Hvolby

*Department of Mathematical Sciences, Aalborg Universitet,
Fredrik Bajers Vej 7G, DK-9220 Aalborg Øst, Denmark
thomas@math.aau.dk*

Frédéric Vrins

*Louvain Finance Center & CORE, Université catholique de Louvain,
Voor du Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium
frederic.vrins@uclouvain.be*

We examine credit value adjustment (CVA) estimation under wrong-way risk (WWR) by computing the expected positive exposure (EPE) under an equivalent measure as suggested in [1], adjusting the drift of the underlying for default risk. We apply this technique to European put and call options and derive the analytic formulas for EPE under WWR obtained with various approximations of the drift adjustment. We give the results of numerical experiments based on 4 parameter sets, and supply figures of the CVA based on both of the suggested proxys, comparing with CVA based on a 2D-Monte Carlo scheme and Gaussian Copula resampling. We also show the CVA obtained by the formulas from Basel III. We observe that the Basel III formula does not account for the credit-market correlation, while the Gaussian Copula resampling method estimates a too large impact of this correlation. The two proxies account for the credit-market correlation, and give results that are mostly similar to the 2D-Monte Carlo results.

Keywords: counterparty risk, CVA, wrong-way risk, change of measure, drift adjustment, wrong way measure, put and call options.

1. Introduction

In this paper we aim at computing the credit valuation adjustment (CVA) expressions of European calls and puts under the Black-Scholes-Merton-Cox

model, that is when the underlying stock follows GBM dynamics and the default is governed by a totally inaccessible stopping time corresponding to the first jump time of a Cox process. Specifically, we assume that the default intensity follows a CIR-process.

Let us consider a portfolio with maturity T and whose discounted price process is $\tilde{V}$. The CVA associated to such a portfolio traded with a counterparty whose recovery rate is R and default time is τ with survival (risk-neutral) probability curve $G(t) := \mathbb{Q}(\tau > t)$ is given by

$$\text{CVA} = -(1 - R) \int_0^T \mathbb{E}^{\mathbb{Q}} \left[\tilde{V}_s^+ \mid \tau = s \right] dG(s), \quad (1)$$

where $x^+ := \max(x, 0)$. It has been shown in [1] that when the default time is modeled as the first jump's time of a Cox process, the “ $\tau = s$ ” condition in the expectation in Eq. (1) — associated to market-credit dependency that is, to wrong-way risk — can be absorbed in the drift of the portfolio price process:

$$\text{CVA} = -(1 - R) \int_0^T \mathbb{E}^{\mathbb{Q}^{C_s}} \left[\tilde{V}_s^+ \right] dG(s).$$

Here, C_t is a rolling numéraire corresponding to the default leg of a CDS offering protection in a small interval around t , and is not to be confused with the call option price at t , noted C_t . We refer the reader to [1] for more details about this technique.

We define the expected positive exposure (EPE) without taking wrong-way risk into account as the expectation in Eq. (1) without the condition. Thus the no-WWR EPE is simply the function $\text{EPE}^\perp(s) := \mathbb{E}^{\mathbb{Q}} \left[\tilde{V}_s^+ \right]$ for $s \in [0, T]$. The EPE under wrong-way risk (referred to as the WWR EPE) is defined as

$$\text{EPE}(s) = \mathbb{E}^{\mathbb{Q}} \left[\tilde{V}_s^+ \mid \tau = s \right] = \mathbb{E}^{\mathbb{Q}^{C_s}} \left[\tilde{V}_s^+ \right].$$

From Girsanov theorem, a $\mathbb{Q}$ -Brownian motion on $[0, s]$ will become, under $\mathbb{Q}^{C_s}$, a Brownian motion plus a drift. In particular, we note θ^s the drift associated to the $\mathbb{Q}$ -Brownian motion driving the exposure. Its analytical expression is derived explicitly in [1]. We now show that when this (stochastic) drift is approximated by a deterministic function $\theta(\cdot, s)$, the quantity $\mathbb{E}^{\mathbb{Q}^{C_s}} \left[\tilde{V}_s^+ \right]$ is available in closed form for calls and puts, leading to an analytical approximation for the CVA under wrong-way risk, and compare the effect on CVA of two approximations of this drift to the Monte Carlo setup.

2. Call and put risk-neutral dynamics

We assume GBM dynamics for the stock under the risk-neutral measure $\mathbb{Q}$, with constant risk-free rate r and volatility $\sigma > 0$. Hence, denote by W a $\mathbb{Q}$ -Brownian motion,

$$dS_t = rS_t dt + \sigma S_t dW_t ,$$

whose solution is

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t} .$$

Let us note C the price process of a European call option on the stock S with maturity T and strike K . Hence, using the Theta-Delta-Gamma relationship,

$$dC_t = \Theta_t dt + \Delta_t dS_t + \frac{1}{2} \Gamma_t d\langle S \rangle_t = rC_t dt + \sigma S_t \Delta_t dW_t ,$$

and it is well-known that

$$\begin{aligned} C_t &= S_t \Phi(d(T-t)) - Ke^{-r(T-t)} \Phi\left(d(T-t) - \sigma\sqrt{T-t}\right) , \\ \Delta_t &= \Phi(d(T-t)) , \\ d(s) &:= \frac{1}{\sigma\sqrt{s}} \left(\ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)s \right) . \end{aligned}$$

Let us note the time- t discounted value of any process $X = (X_t)_{t \geq 0}$ as $\tilde{X}_t := X_t e^{-rt}$. The discounted call price process $\tilde{C}$ can be written in terms of the discounted stock price process $\tilde{S}$:

$$\begin{aligned} \tilde{C}_t &= \tilde{S}_t \Phi(d(t, T)) - Ke^{-rT} \Phi\left(d(t, T) - \sigma\sqrt{T-t}\right) , \\ d(t, T) &= \frac{1}{\sigma\sqrt{T-t}} \left(\ln \frac{\tilde{S}_t}{K} + rT + \frac{\sigma^2}{2}(T-t) \right) , \end{aligned}$$

where we have used that $\ln S_t = \ln \tilde{S}_t + rt$. Using $W_t \stackrel{(\mathbb{Q})}{\sim} \sqrt{t}Z$ where $Z \stackrel{(\mathbb{Q})}{\sim} \mathcal{N}(0, 1)$, one obtains

$$\begin{aligned} \tilde{S}_t &= S_0 e^{-\frac{\sigma^2}{2}t + \sigma W_t} \sim S_0 e^{-\frac{\sigma^2}{2}t + \sigma\sqrt{t}Z} , \\ \Delta_t &= \Phi\left(\frac{1}{\sigma\sqrt{T-t}} \left(\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T - \sigma^2 t \right) + \frac{W_t}{\sqrt{T-t}}\right) \\ &\sim \Phi\left(\underbrace{\frac{1}{\sigma\sqrt{T-t}} \left(\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T - \sigma^2 t \right)}_{:=\alpha(t)} + \underbrace{\frac{\sqrt{t}}{\sqrt{T-t}} Z}_{:=\beta(t)}\right) , \end{aligned}$$

so that

$$\begin{aligned}\tilde{C}_t &= S_0 e^{-\frac{\sigma^2}{2}t + \sigma W_t} \Phi\left(\alpha(t) + W_t/\sqrt{T-t}\right) \\ &\quad - K e^{-rT} \Phi\left(\alpha(t) - \sigma\sqrt{T-t} + W_t/\sqrt{T-t}\right) \\ &\sim S_0 e^{-\frac{\sigma^2}{2}t + \sigma\sqrt{t}Z} \Phi\left(\alpha(t) + \beta(t)Z\right) \\ &\quad - K e^{-rT} \Phi\left(\alpha(t) - \sigma\sqrt{T-t} + \beta(t)Z\right) .\end{aligned}$$

A similar development yields the dynamics and the marginal distributions of the corresponding put

$$\begin{aligned}\tilde{P}_t &= K e^{-rT} \Phi\left(\sigma\sqrt{T-t} - \alpha(t) - W_t/\sqrt{T-t}\right) \\ &\quad - S_0 e^{-\frac{\sigma^2}{2}t + \sigma W_t} \Phi\left(-\alpha(t) - W_t/\sqrt{T-t}\right) \\ &\sim K e^{-rT} \Phi\left(\sigma\sqrt{T-t} - \alpha(t) - \beta(t)Z\right) \\ &\quad - S_0 e^{-\frac{\sigma^2}{2}t + \sigma\sqrt{t}Z} \Phi\left(-\alpha(t) - \beta(t)Z\right) .\end{aligned}$$

3. Expected positive exposures under no WWR

As $\tilde{C} \geq 0$ and $\tilde{P} \geq 0$, the expected (discounted) exposure corresponds to the expected *positive* (discounted) exposure. Hence,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\tilde{C}_t] &= S_0 e^{-\frac{\sigma^2}{2}t} \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{t}Z} \Phi\left(\alpha(t) + \beta(t)Z\right)\right] \\ &\quad - K e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\alpha(t) - \sigma\sqrt{T-t} + \beta(t)Z\right)\right] \\ &= S_0 \Phi\left(\frac{\alpha(t) + \beta(t)\sigma\sqrt{t}}{\sqrt{1 + \beta^2(t)}}\right) - K e^{-rT} \Phi\left(\frac{\alpha(t) - \sigma\sqrt{T-t}}{\sqrt{1 + \beta^2(t)}}\right) , \\ \mathbb{E}^{\mathbb{Q}}[\tilde{P}_t] &= K e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\sigma\sqrt{T-t} - \alpha(t) - \beta(t)Z\right)\right] \\ &\quad - S_0 e^{-\frac{\sigma^2}{2}t} \mathbb{E}^{\mathbb{Q}}\left[e^{-\sigma\sqrt{t}Z} \Phi\left(-\alpha(t) - \beta(t)Z\right)\right] \\ &= K e^{-rT} \Phi\left(\frac{\sigma\sqrt{T-t} - \alpha(t)}{\sqrt{1 + \beta^2(t)}}\right) - S_0 \Phi\left(\frac{-\alpha(t) - \beta(t)\sigma\sqrt{t}}{\sqrt{1 + \beta^2(t)}}\right) ,\end{aligned}$$

where we have used

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\eta Z} \Phi\left(\mu + \sigma Z\right)\right] = e^{\frac{\eta^2}{2}} \Phi\left(\frac{\mu + \sigma\eta}{\sqrt{1 + \sigma^2}}\right) .$$

It can be checked that $\mathbb{E}^{\mathbb{Q}}[\tilde{C}_t] = C_0$ and $\mathbb{E}^{\mathbb{Q}}[\tilde{P}_t] = P_0$ for all $t \in [0, T]$ as expected from the martingale property of discounted price processes under

$\mathbb{Q}$. Nevertheless, because of the drift-adjustment, those expressions will become time-dependent as soon as WWR will enter the picture.

4. Expected positive exposures under WWR

Under no WWR (risk-neutral measure $\mathbb{Q}$), $\tilde{C}$ is a martingale,

$$d\tilde{C}_t = \sigma \tilde{S}_t \Delta_t dW_t ,$$

whose solution is given by the standard Black-Scholes-Merton equation in Sec. 3. As discussed above, Girsanov theorem yields

$$dW_t = dW_t^s + \theta_t^s dt ,$$

where W^s is a $\mathbb{Q}^{\mathcal{C}_s}$ -Brownian motion on $[0, s]$. We assume that under $\mathbb{Q}$, the default intensity λ is governed by a CIR process with volatility η , i.e.

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \eta\sqrt{\lambda_t}dW_t^\lambda ,$$

where W_t^λ is a $\mathbb{Q}$ -Brownian motion whose correlation with W is ρ . A non-zero value for ρ introduces a dependency between S and λ that controls wrong-way risk. The drift adjustment is given by [1]

$$\theta_t^s = \theta_t^s(\lambda_t) = \rho\eta\sqrt{\lambda_t} \left(\frac{A^\lambda(t, s)B_s^\lambda(t, s)}{A^\lambda(t, s)B_s^\lambda(t, s)\lambda_t - A_s^\lambda(t, s)} - B^\lambda(t, s) \right) , \quad (2)$$

where A^λ, B^λ are known zero-coupon bond functions in affine models [2]:

$$\mathbb{E}^\mathbb{Q} \left[e^{-\int_t^s \lambda_u du} \middle| \mathcal{F}_t \right] = A^\lambda(t, s)e^{-B^\lambda(t, s)\lambda_t} .$$

The subscripts refer to the variable with respect to which we compute the derivatives of A^λ and B^λ .

Let us now look at the dynamics of the call for $t \in [0, s]$ under $\mathbb{Q}^{\mathcal{C}_s}$. First, observe that we can write $\tilde{C}$ as a deterministic function of the variables (t, W_t) (instead of the usual (t, S_t) couple):

$$\tilde{C}_t = v(t, W_t) ,$$

with

$$\begin{aligned} v(s, x) := & S_0 e^{-\frac{\sigma^2}{2}s + \sigma x} \Phi \left(\alpha(s) + x/\sqrt{T-s} \right) \\ & - K e^{-rT} \Phi \left(\alpha(s) + x/\sqrt{T-s} - \sigma\sqrt{T-s} \right) . \end{aligned}$$

Applying Ito's lemma,

$$d\tilde{C}_t = \left(v_t(t, W_t) + \frac{1}{2}v_{xx}(t, W_t) \right) dt + v_x(t, W_t)dW_t ,$$

and we have, for all (s, x) where $s \in [0, T]$ and $x \in \mathbb{R}$ the following relationships for the partial derivatives of v :

$$v_t(s, x) + \frac{1}{2}v_{xx}(s, x) = 0 \quad (3)$$

$$v_x(s, x) = \sigma S_0 e^{-\frac{\sigma^2}{2}s + \sigma x} \Phi\left(\alpha(s) + x/\sqrt{T-s}\right). \quad (4)$$

Now let us look at the dynamics of the call as a function of the $\mathbb{Q}^{\mathcal{C}_s}$ -Brownian motion W^s on $t \in [0, s]$:

$$\tilde{C}_t := v\left(t, W_t^s + \int_0^t \theta_u^s du\right).$$

Using Ito's lemma and the relationships between v_t, v_x and v_{xx} in Eqs. (3) and (4), we have

$$\begin{aligned} d\tilde{C}_t &= \left(v_t\left(t, W_t^s + \int_0^t \theta_u^s du\right) + \frac{1}{2}v_{xx}\left(t, W_t^s + \int_0^t \theta_u^s du\right) \right) dt \\ &\quad + v_x\left(t, W_t^s + \int_0^t \theta_u^s du\right) (dW_t^s + \theta_t^s dt) \\ &= v_x\left(t, W_t^s + \int_0^t \theta_u^s du\right) \theta_t^s dt + v_x\left(t, W_t^s + \int_0^t \theta_u^s du\right) dW_t^s. \end{aligned}$$

Defining now

$$\begin{aligned} \hat{S}_t &:= \tilde{S}_t e^{\sigma \int_0^t \theta_u^s du}, \\ \hat{\Delta}_t &:= \Phi\left(\hat{d}(t, T)\right), \\ \hat{d}(t, T) &:= \frac{1}{\sigma\sqrt{T-t}} \left(\ln \frac{\hat{S}_t}{K} + rT + \frac{\sigma^2}{2}(T-t) \right), \end{aligned}$$

one gets

$$\begin{aligned} d\tilde{C}_t &= \sigma \tilde{S}_t e^{\sigma \int_0^t \theta_u^s du} \hat{\Delta}_t \theta_t^s dt + \sigma \tilde{S}_t e^{\sigma \int_0^t \theta_u^s du} \hat{\Delta}_t dW_t^s \\ &= \sigma \hat{S}_t \hat{\Delta}_t \theta_t^s dt + \sigma \hat{S}_t \hat{\Delta}_t dW_t^s. \end{aligned} \quad (5)$$

Clearly, $\tilde{C}$ is a $\mathbb{Q}$ -martingale. This is no longer true under the new measure: it features a drift. Moreover, the martingale part is impacted by the drift as well as $\hat{S}$ features θ .

Let us consider the deterministic approximation $\theta_t^s \approx \theta(t, s)$ where λ_t is replaced by a deterministic proxy $\lambda(t)$. By replacing λ_t with $\lambda(t)$ in Eq. (2), we have

$$\theta(t, s) := \rho\eta\sqrt{\lambda(t)} \left(\frac{A^\lambda(t, s)B_s^\lambda(t, s)}{A^\lambda(t, s)B_s^\lambda(t, s)\lambda(t) - A_s^\lambda(t, s)} - B^\lambda(t, s) \right). \quad (6)$$

Then, the WWR EPE expression $\text{EPE}(s) = \mathbb{E}^{\mathbb{Q}^{C^s}} [\tilde{C}_s]$ is known analytically. To compute $\text{EPE}(t)$, the WWR EPE at time t , we need to evaluate the expectation of $\tilde{C}_t$ under $\mathbb{Q}^{C^t}$. We thus set $s = t$ and define

$$\begin{aligned}\Theta(t) &:= \int_0^t \theta(u, t) du, \\ \hat{\alpha}(t) &:= \alpha(t) + \frac{\Theta(t)}{\sqrt{T-t}}, \\ Z^t &\stackrel{(\mathbb{Q}^{C^t})}{\sim} \mathcal{N}(0, 1).\end{aligned}$$

This yields (up to the approximation of the stochastic drift by its deterministic expression)

$$\begin{aligned}\tilde{C}_t &= v(t, W_t^t + \Theta(t)) \\ &= S_0 e^{-\frac{\sigma^2}{2}t + \sigma\Theta(t) + \sigma W_t^t} \Phi\left(\hat{\alpha}(t) + W_t^t / \sqrt{T-t}\right) \\ &\quad - K e^{-rT} \Phi\left(\hat{\alpha}(t) - \sigma\sqrt{T-t} + W_t^t / \sqrt{T-t}\right) \\ &\sim S_0 e^{-\frac{\sigma^2}{2}t + \sigma\Theta(t) + \sigma\sqrt{t}Z^t} \Phi\left(\hat{\alpha}(t) + \beta(t)Z^t\right) \\ &\quad - K e^{-rT} \Phi\left(\hat{\alpha}(t) - \sigma\sqrt{T-t} + \beta(t)Z^t\right),\end{aligned}$$

showing that the WWR EPE takes a similar form as the No-WWR EPE:

$$\mathbb{E}^{\mathbb{Q}^{C^t}} [\tilde{C}_t] \approx S_0 e^{\sigma\Theta(t)} \Phi\left(\frac{\hat{\alpha}(t) + \beta(t)\sigma\sqrt{t}}{\sqrt{1 + \beta^2(t)}}\right) - K e^{-rT} \Phi\left(\frac{\hat{\alpha}(t) - \sigma\sqrt{T-t}}{\sqrt{1 + \beta^2(t)}}\right),$$

where the approximation results from the fact that we have replaced the random variable $\int_0^t \theta_u^t du$ by the deterministic quantity $\int_0^t \theta(u, t) du$. As regards to the WWR EPE of the put, one easily gets

$$\mathbb{E}^{\mathbb{Q}^{C^t}} [\tilde{P}_t] \approx K e^{-rT} \Phi\left(\frac{\sigma\sqrt{T-t} - \hat{\alpha}(t)}{\sqrt{1 + \beta^2(t)}}\right) - S_0 e^{\sigma\Theta(t)} \Phi\left(\frac{-\hat{\alpha}(t) - \beta(t)\sigma\sqrt{t}}{\sqrt{1 + \beta^2(t)}}\right).$$

5. Proxys of θ_t^s

Here we use two different proxys for θ_t^s . As presented in [1], we consider a proxy where the $\mathbb{Q}$ -expectation of λ_t is used in the formula for the drift adjustment (6). However, here we also present an alternative proxy, by using an approximation of the $\mathbb{Q}^{C^T}$ -expectation of λ_t and compare the impact on CVA in Sec. 7.

5.1. $\mathbb{Q}$ -expectation

Here we use $\theta(t, s) = \theta_t^s(\bar{\lambda}_t)$, where $\bar{\lambda}_t := \mathbb{E}^{\mathbb{Q}}[\lambda_t]$. One strength for this proxy is that we have an analytic formula for $\bar{\lambda}_t$ and the proxy $\theta(t, s)$ is straight-forward to obtain. The disadvantage is that we are ‘operating’ under other measures than $\mathbb{Q}$. Specifically when estimating the WWR EPE at time t , we have changed measure to $\mathbb{Q}^{C_t}$. This changes the dynamics of λ , but it is ignored in this approach.

5.2. Approximation of $\mathbb{Q}^{C_T}$ -expectation

In order to improve the deterministic approximation of θ_t^s , we aim to obtain an approximation for the $\mathbb{Q}^{C_T}$ -expectation of λ_t . Remark that we use the measure for the maturity of the contract for all $t \in [0, T]$. A possible weakness of this proxy is that for calculating WWR EPE at time t we should use the $\mathbb{Q}^{C_t}$ -dynamics not the (terminal) $\mathbb{Q}^{C_T}$ -dynamics. However, using the terminal measure is a more convenient choice, since it is just necessary to obtain one ‘term structure’ of λ_t , $t \in [0, T]$, whereas using the $\mathbb{Q}^{C_t}$ -dynamics for the WWR EPE at time t will have the effect that it is necessary to compute separate values for λ_u , $u \in [0, t]$, corresponding to each $t \in [0, T]$. This may be computationally heavy, and thus we assume the simpler version with the benefit that only one term structure has to be computed while the effect of the drift-adjustment in λ from the change of measure may be accounted for. A closed-form expectation of λ_t under $\mathbb{Q}^{C_T}$ can however not be readily found, but in the following we present an approximation of this expectation.

One further remark is that the $\mathbb{Q}^{C_T}$ -dynamics of λ is completely independent of the correlation between the underlying stock and λ , but is solely determined by the parameters of the $\mathbb{Q}$ -dynamics of λ as well as the maturity of the contract. This allows for computed λ ’s to be used for calculating CVA on several contracts with the same counterparty, since the dynamics of the contract and its correlation with the default intensity does not enter any of the expressions.

Firstly, consider the $\mathbb{Q}^{C_T}$ -dynamics of λ_t for $t \in [0, T]$:

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \eta\sqrt{\lambda_t} \left(dW_t^T + \eta\sqrt{\lambda_t}\tilde{\theta}_t^T(\lambda_t)dt \right), \quad (7)$$

$$\tilde{\theta}_t^T(x) := \frac{a(t, T)}{a(t, T)x - b(t, T)} - c(t, T),$$

where

$$a(t, T) := A^\lambda(t, s)B_s^\lambda(t, s) \quad , \quad b(t, T) := A_s^\lambda(t, s) \quad \text{and} \quad c(t, T) := B^\lambda(t, s) \quad .$$

Hence, integrating both sides of the above SDE in Eq. (7),

$$\lambda_t = \lambda_0 + \kappa\theta t - \kappa \int_0^t \lambda_s ds + \eta^2 \int_0^t \lambda_s \tilde{\theta}_s^T(\lambda_s) ds + \eta \int_0^t \sqrt{\lambda_s} dW_s^T$$

and using Tonelli's theorem,

$$\mathbb{E}^{\mathbb{Q}^{C^T}}[\lambda_t] = \lambda_0 + \kappa\theta t - \kappa \int_0^t \mathbb{E}^{\mathbb{Q}^{C^T}}[\lambda_s] ds + \eta^2 \int_0^t \mathbb{E}^{\mathbb{Q}^{C^T}}[\lambda_s \tilde{\theta}_s^T(\lambda_s)] ds, \quad (8)$$

where the Ito integral has zero expectation, and thus has vanished. We want to simplify the term that includes

$$\lambda_s \tilde{\theta}_s^T(\lambda_s) = \frac{a(t, T)\lambda_s}{a(t, T)\lambda_s - b(t, T)} - c(t, T)\lambda_s,$$

and therefore we use a first-order Taylor-expansion of the function $a(t, T)x/(a(t, T)x - b(t, T))$ around some point $x(t) > 0$. We use a positive function since $\mathbb{Q}^{C^T}$ is an equivalent measure to $\mathbb{Q}$ and thus the expectation of λ_t is always positive. Expanding the function around zero also turns out to be an undesirable choice that leads to unstable estimates of the expectation close to maturity, as $a(t, T)/b(t, T)$ diverges for $t \rightarrow T$. We choose to make the expansion around the $\mathbb{Q}$ -expectation of λ_t , since this is indeed a positive function, and the $\mathbb{Q}$ -expectation may give some reasonable input to the $\mathbb{Q}^{C^T}$ -expectation. The Taylor expansion looks as follows

$$\begin{aligned} \frac{a(t, T)x}{a(t, T)x - b(t, T)} &= \frac{a(t, T)x(t)}{a(t, T)x(t) - b(t, T)} \\ &\quad - \frac{a(t, T)b(t, T)}{(a(t, T)x(t) - b(t, T))^2} (x - x(t)) + o(x) \\ &= \left(\frac{a(t, T)x(t)}{a(t, T)x(t) - b(t, T)} \right)^2 - \frac{a(t, T)b(t, T)}{(a(t, T)x(t) - b(t, T))^2} x \\ &\quad + o(x). \end{aligned}$$

Setting $g(t) := \mathbb{E}^{\mathbb{Q}^{C^T}}[\lambda_t]$ we have

$$\begin{aligned} g(t) &\approx \lambda_0 + \kappa\theta t - \kappa \int_0^t g(s) ds \\ &\quad - \eta^2 \int_0^t \left(\frac{a(s, T)b(s, T)}{(a(s, T)x(s) - b(s, T))^2} + c(s, T) \right) g(s) ds \\ &\quad + \eta^2 \int_0^t \left(\frac{a(s, T)x(s)}{a(s, T)x(s) - b(s, T)} \right)^2 ds. \end{aligned}$$

Differentiating both sides we obtain a first-order linear inhomogeneous ODE

$$g'(t) \approx \kappa\theta + \eta^2 \left(\frac{a(t, T)x(t)}{a(t, T)x(t) - b(t, T)} \right)^2 - h(t, T)g(t) ,$$

where

$$h(t, T) := \kappa + \eta^2 \left(\frac{a(t, T)b(t, T)}{(a(t, T)x(t) - b(t, T))^2} + c(t, T) \right) .$$

Disregarding the drift approximation the solution to this SDE is

$$g(t) = e^{-H(t, T)} \left(g(0) + \int_0^t (\kappa\theta + \eta^2 G(s, T)) e^{H(s, T)} ds \right) ,$$

where in this context, $g(0) = \lambda_0$, and

$$H(s, T) := \int_0^s h(u, T) du ,$$

$$G(s, T) := \left(\frac{a(s, T)x(s)}{a(s, T)x(s) - b(s, T)} \right)^2 .$$

6. Potential future exposures (PFE)

We would like to compare the risk-neutral CVA (CVA computed with market-implied default probabilities and WWR EPE) with actuarial CVA, computed with PFE (e.g. 99% quantile of exposures) and historical default probabilities.

From above, the time- t call price takes the form $C_t = f(t, W_t)$ where

$$f(t, x) = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma x} \Phi \left(\alpha(t) + x/\sqrt{T-t} \right) - K e^{-r(T-t)} \Phi \left(\alpha(t) - \sigma\sqrt{T-t} + x/\sqrt{T-t} \right)$$

is a continuous increasing function. Under the risk-neutral measure, the k -PFE is defined as the profile of the exposure's quantile at level k : Functions f being continuous and strictly increasing wrt x , this means that

$$q(t) = f \left(t, \Phi^{-1}(k)\sqrt{t} \right) .$$

A similar expression holds for WWR PFE: one just needs to account for the measure change. This can be achieved by inserting the drift adjustment, replacing $f(t, x)$ by $f(t, x + \Theta(t))$ so that $q(t) = f(t, \Phi^{-1}(k)\sqrt{t} + \Theta(t))$.

The move from “risk-neutral” to “historical” PFEs can be handled in a similar way. Clearly, C_t is a deterministic function of S_t , so that the

historical distribution of C_t is obtained by using μ (instead of r) as rate of return of S on the period $[0, t)$. Because S_t is a deterministic function of W_t and $W_t = \tilde{W}_t + \frac{\mu-r}{\sigma}t$ where $\tilde{W}$ is a Brownian motion under the physical measure, the “historical quantile” is obtained by evaluating the “risk-neutral” function f at a shifted point compared to the Normal quantile: $q(t) = f\left(t, \Phi^{-1}(k)\sqrt{t} + \frac{\mu-r}{\sigma}t\right)$, and similarly for the WWR case.

Notice that in the previous sections — when dealing with CVA based on EPE — we assume that the specified CIR-process is the true risk-neutral dynamics of the default intensities. Thus we do not consider fitting the survival probability curve to any exogenously given CDS quotes and to obtain the CVA from Eq. (1) we apply $G(t) = P^{\text{CIR}}(0, t)$ where P^{CIR} is the CIR-bond price.

This approach needs adjustment to obtain the actuarial CVA part. Here we assign a rating and a flat equivalent spread (or hazard rate) γ to each pair of maturity and parameter set. We apply a historical survival probability curve based on the rating: $G(t) = \exp\{-\gamma t\}$. To ensure that we can recover the historical probability curve when applying the intensities process we use the CIR++ model, see e.g. [2], and apply deterministic shifts $\phi(t)$ such that the shifted intensity process $y_t = \lambda_t + \phi(t)$ is inline with the flat hazard rate γ . The survival probabilities implied by the shifted intensity process are found by

$$P^y(t, s) := \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^s y_u du} \right] = e^{-\int_t^s \phi(u) du} P^{\text{CIR}}(t, s) \quad , \quad s \geq t.$$

In the approximation of the actuarial CVA based on the drift adjustments, the shifted intensities y_t replace λ_t in Eq. (2). Further we choose $\phi(0) = 0$ such that $y_0 = \lambda_0$ and replace $A(t, s)$ by $A^*(t, s) = \exp\{-\int_t^s \phi(u) du\} A(t, s)$, while $B(t, s)$ remains.

7. Numerical experiments

We use the four parameter sets for the CIR-process of λ used in [1]. Thus in the forthcoming we will regard parameters in Table 1 as set 1–4. Further we use $S_0 = K = 15$, $r = 1\%$, $\mu = 3\%$ and $\sigma = 30\%$, and call options with a time to maturity of 1 year and 5 years, respectively. For simplicity we assume that there is no recovery upon default, i.e. $R = 0$ in Eq. (1). The corresponding CVA figures are given in Figs. 1 and 2, and are compared with a 2D Monte Carlo scheme as well as the Gaussian Copula resampling approach (for more details about this approach, see e.g. [3], [4] or [5]).

Table 1. Parameter sets for the dynamics of λ in the numerical experiments.

	λ_0 (bps)	κ	θ (bps)	η	Rating used for actuarial CVA
Set 1	300	02%	1610	8%	B
Set 2	350	35%	450	15%	B
Set 3	100	80%	200	20%	B
Set 4	300	50%	500	50%	B

We also consider the actuarial CVA; CVA calculated from the 99% PFE, which is described in Sec. 6, and on historical rather than risk-neutral default probabilities. We use the default rates from [6], and we consider all parameter sets to have rating *B*, which ensures that all shifts are positive and thus the shifted default intensity is ensured to be non-negative. Specifically the flat hazard rates used for *B* rating are 458.8 bps for 1 year contracts and 507.6 bps for 5 year contracts. The actuarial CVA is shown in Figs. 3 and 4.

In the following, we use the terms $\mathbb{Q}^{C^T}$ -proxy and $\mathbb{Q}$ -proxy for the proxys of θ_t^s using the λ expectation under $\mathbb{Q}^{C^T}$ and $\mathbb{Q}$, respectively.

Consider the CVA on 1 year contracts in Fig. 1. We observe a pattern that for negative correlations, we tend to estimate a higher CVA compared to the 2D Monte Carlo scheme. Generally this overestimation of CVA is stronger for the $\mathbb{Q}^{C^T}$ -proxy than for the $\mathbb{Q}$ -proxy. The exception is parameter set 4, where the $\mathbb{Q}$ -proxy which estimates a slightly lower CVA. For positive correlations — when we experience WWR on the call — we observe a very good fit of the $\mathbb{Q}^{C^T}$ -proxy and the Monte Carlo CVAs parameter set 1–3, while the $\mathbb{Q}$ -proxy also tends to overestimate the CVA on this end.

Comparing the results for the two proxys, we observe that the $\mathbb{Q}$ -proxy tends to suggest a larger WWR-effect, giving a larger compensation than the $\mathbb{Q}^{C^T}$ -proxy for positive correlations, while it suggests a lower compensation in the case of negative correlations. Thus the $\mathbb{Q}$ -proxy suggests a higher impact of the “market-credit correlation”.

Consider now the CVA on the 5 year contract in Fig. 2. Firstly, we observe that even for one million sample paths and a time step of 0.01, the Monte Carlo simulations of the CVA does include some bias, since the zero-correlation case does not completely correspond with the analytic formula. This is especially pronounced for parameter set 4. Here we experience a weakness of the 2D Monte Carlo approach; it is computationally heavy, but

Fig. 1. 1 year call CVA profiles for the four parameter sets using a 2D Monte Carlo scheme (line with circles) with 10^6 paths and a time step of 0.01, compared with the analytic approximation using the Q -expectation (black) and Q^{CT} -approximation (grey). CVA based on the Gaussian Copula resampling approach (line with squares) and the analytic CVA with zero-correlation (vertical dashed grey line) are included. The Basel no-WWR CVA is indicated by the line with triangles and the WWR CVA using $\alpha = 1.4$ is indicated by grey dots.

Fig. 2. 5 year call CVA profiles for the four parameter sets using a 2D Monte Carlo scheme (line with circles) with 10^6 paths and a time step of 0.01, compared with the analytic approximation using the $\mathbb{Q}$ -expectation (black) and $\mathbb{Q}^{CT}$ -approximation (grey). CVA based on the Gaussian Copula resampling approach (line with squares) and the analytic CVA with zero-correlation (dashed grey line) are included. The Basel no-WWR CVA is indicated by line with triangles and the WWR CVA using $\alpha = 1.4$ is indicated by grey dots.

moreover includes a bias with a very small time-step and large number of paths. Comparing the CVA of the two proxys, we observe similar behavior as in the 1 year case.

For both 1 and 5 year CVA — and for all parameter sets — we observe that the resampling approach is highly sensitive to the market-credit correlation. This allows to model very strong WWR impact. The problem however is that it is unclear how the value of the dependence parameter (the correlation between τ and the exposure V_{t_i} at any given point in time t_i) has to be chosen.

Further, we provide the CVA under Basel, based on [7], [8]. In this context (T being the contract maturity, where we consider 1 and 5 years), the No-WWR figure is given by

$$\text{CVA}^{\text{basel}} = (1 - G(T)) \frac{\text{EPE}^\perp(0) + \text{EPE}^\perp(T)}{2},$$

where $\text{EPE}^\perp(t)$ is the (No-WWR) EPE at time t , which is constant and equal to $\tilde{C}_0$ for the call and $\tilde{P}_0$ for the put. The WWR CVA is given by

$$\text{CVA}_\alpha^{\text{basel}} = (1 - G(T)) \alpha \text{EPE}^\perp(T/2), \quad (9)$$

where $\alpha \text{EPE}^\perp(T/2)$ is called the “exposure at default” (EAD) and the scaling coefficient α is typically set to 1.4. The corresponding levels are indicated on Figs. 1 and 2.

The Basel III parameter α cannot be considered as a way to represent market-credit correlation, but in fact capture the “market-credit covariance”. This is a crucial point, that complicates drastically the choice of a reasonable value for α . In order for the Basel type formula to be a decent approach to account for WWR, α has to be chosen not only with regards to dependence between portfolio and credit, but also according to both market and credit volatilities. This observation suggests that it is a bit naive to hope that a kind of “universal constant” would be able to account for this effect. The approximation proposed in [1] is therefore, from this perspective, a significant improvement to Basel type formulae. Table 2 show the values of α that make the CVA in the Basel type formulae agree with the ones obtained from Monte Carlo simulation for 1 year contracts. Across parameter sets the value changes quite significantly. Obviously $\alpha \approx 1$ for zero-correlation, and the values are larger (smaller) for positive (negative) correlations. Thus the Basel approach cannot capture right-way risk, which is experienced when the correlation is negative, while the performance of the method is highly dependent both on the correlation and the parameter set used.

Table 2. The values of α for which Eq. (9) agrees with the 2D-MC results for 1 year call options.

ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
Set 1	0.75	0.83	0.92	1.01	1.10	1.20	1.30
Set 2	0.63	0.74	0.87	1.01	1.16	1.31	1.48
Set 3	0.42	0.58	0.78	1.01	1.28	1.59	1.95
Set 4	0.29	0.45	0.70	1.03	1.44	1.95	2.57

The interpretation of the actuarial CVA in Figs. 3 and 4 are similar to the interpretation of the risk-neutral CVA results. We generally observe a larger impact of the correlation on the CVA when using the $\mathbb{Q}$ -proxy than what suggested by the $\mathbb{Q}^{C_T}$ -proxy.

8. Conclusion

From the change-of-measure approach suggested in [1], we have examined the CVA on put and call options under WWR. In the Basel III framework, WWR is treated by a multiplier. But it should not be based on “market-credit correlation”, but rather on “market-credit covariance”. We find that the Basel III approach is a naive way of estimating the CVA that does not recognize right-way risk and cannot capture the wrong-way risk in a desirable way. However, using the set-up in this paper, one can obtain CVA on put and call options, and capture the effect of the market-credit correlation by analytic formulas. Specifically, we present the formulas for two proxys of the drift-adjustment process, using each proxy, the CVA can be obtained analytically. Further the actuarial CVA based on PFE also has an analytic expression, based on the formulas in the paper.

In the numerical experiments, we examine the estimated CVA — both the risk-neutral and actuarial CVA — from the formulas supplied in the paper, compared with joint (exposure-credit) Monte Carlo simulations, Gaussian Copula resampling and Basel III figures. We observe that the Gaussian Copula resampling approach is very sensitive to the correlation, leading to too high CVA estimates when experiencing WWR. We do not find 2D Monte Carlo to be a desirable method, since it is computationally heavy, and in some cases includes a bias, even for a small time-step and large number of sample paths.

On the other hand, we get very encouraging results from the CVA based on the two proxys, both when calculating the risk-neutral CVA and the actuarial CVA. The simple $\mathbb{Q}$ -proxy is performing quite reasonably and

Fig. 3. 1 year actuarial call CVA profiles for the four parameter sets using the analytic approximation with the $\mathbb{Q}$ -expectation (solid line) and $\mathbb{Q}^{CT}$ -approximation (dashed line). The dashed grey line shows the analytic CVA with zero-correlation is included.

Fig. 4. 5 year actuarial call CVA profiles for the four parameter sets using the analytic approximation with the $\mathbb{Q}$ -expectation (solid line) and $\mathbb{Q}^{CT}$ -approximation (dashed line). The dashed grey line shows the analytic CVA with zero-correlation is included.

captures the general behavior of the CVA. This proxy is very easy to implement and fast to compute since all formulas are analytic (up to the deterministic approximation of the drift adjustment). Using the $\mathbb{Q}^{\mathcal{C}_T}$ -proxy requires an additional approximation (to compute the corresponding expectation of λ_t), but gives a slightly more realistic (considering the MC-results to be the true CVA) CVA when experiencing WWR.

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