An Introduction to the
Mathematical Structure

of
Quantum Mechanics

A Short Course for Mathematicians

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An Introduction to the
Mathematical Structure
— *of* —
Quantum Mechanics

A Short Course for Mathematicians

Second Edition

Lecture Notes by **F. Strocchi**

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Preface

These notes were written for a short series of lectures at the Scuola Normale Superiore, Pisa, in the academic years 1993-94, 1994-95, 1995-96, in order to fulfill a request by the mathematicians to have a short (half-semester) course for mathematical students willing to learn the main ideas and mathematical structure of Quantum Mechanics.

Since the books on Quantum Mechanics are hundreds and some of them very good, also for a mathematically minded student, one could reasonably question the need of having another one. The main point is that, contrary to Classical Mechanics, Quantum Mechanics is not part of the common education of mathematical students, even if this theory is at the roots of important developments of modern mathematics, as briefly discussed in the Introduction. Perhaps one of the reasons is that most of the books on Quantum Mechanics start from emphasizing and discussing the many experimental facts, which led to the crisis of Classical Mechanics, whose appreciation require a non-superficial familiarity with classical physics, with the result that the tight logical and mathematical structure of Quantum Mechanics is usually relegated to a secondary level. In particular, the usual presentation, which assumes that states are described by vectors of a Hilbert space (and observables by Hilbert space operators) on the basis of the so-called superposition principle, may not be appealing to a mathematical student. On the other hand, the books on the mathematical foundations of Quantum Mechanics, above all the beautiful books by J. von Neumann, G. Mackey, J.M. Jauch, C. Piron, do not seem to have the introductory character suitable for undergraduate students.

Moreover, the C^* -algebraic formulation of Quantum Mechanics, which has unquestionable advantages for logic and conceptual economy, especially for a mathematically oriented audience, and has played a crucial role for the recent non-commutative extensions of Calculus, Geometry, Probability etc., has not yet become standard in quantum mechanics textbooks.

The aim of these notes is to introduce the mathematical student to the basic mathematical structure and foundations of Quantum Mechanics, by emphasizing as a starting point the C^* -algebraic structure of the algebra of observables, which defines a physical system.

An effort is made in motivating and possibly simplifying Segal's postulates for Quantum Mechanics; the C^* -algebraic structure of the algebra generated by the observables is argued to follow essentially from simple basic physical ideas, by exploiting the operational definition of measurements and the duality between states and observables.

Once it is accepted that the observables generate a C^* -algebra, a property which is also implied (actually in a stronger form) by one of the Dirac-von Neumann axioms, the mathematical structure of Quantum Mechanics, namely the description of the physical states by Hilbert space vectors and the representation of the observables by Hilbert space operators, follows from the GNS and Gelfand-Naimark theorems; moreover, for quantum mechanical systems with a *finite* number of degrees of freedom, Schrödinger Quantum Mechanics is derived as the direct consequence of Heisenberg uncertainty relations (leading to Heisenberg commutation relations) and the von Neumann uniqueness theorem.

No pretension of completeness is made about the subject; the scope is that of an introductory short course on the basic knowledge of Quantum Mechanics which should hopefully become part of the common education also of mathematicians. A few basic examples of quantum systems, *in primis* the hydrogen atom, are worked out with attention to the mathematical setting and logic.

An effort is made in simplifying the technical points and the proofs, especially in the short account of C^* -algebra theory, and in reducing the background knowledge needed to the minimum; the basic notions of probability theory and of Hilbert space operators are required.

The chapters or paragraphs marked with an asterisk were not part of the short course and were discussed in additional lectures; they are included here for completeness and may be skipped in a first reading. The increasing interest on the functional integral approach to quantum mechanics and more generally the deep connection between quantum mechanics and stochastic processes has motivated the brief account given in Chapter 6, with emphasis on the main ideas and the basic mathematical structures.

The effectiveness of the functional integral approach for both the solution of quantum mechanical problems and the analysis of general properties is briefly discussed in Chapter 6; in particular its relevance for Feynman perturbative expansion, the classical limit, the interplay between quantization and topology, the discussion of coupling constant analyticity are worked out in some simple, but illuminating cases.

Since the C^* -algebraic approach has proved inevitable for the mathematical description of quantum systems with infinite degrees of freedom, and useful for the discussion of deep theoretical physics problems, like spontaneous symmetry breaking, phase transitions, quantum statistical mechanics, quantum field theory etc., the lectures might turn out to be useful also for theoretical physics students, who might also profit from a look to

the functional integral approach beyond the heuristic presentations usually adopted in theoretical physics books, especially in view of its use for a non-perturbative approach to quantum field theory and many body theory.

I am grateful to the *Accademia dei Lincei* and the University of Rome “*La Sapienza*” for inviting me to give a series of lectures and for the opportunity of confronting this presentation of Quantum Mechanics with a very selected audience. The preparation of these lectures notes has benefited from enlightening discussions and collaboration with G. Morchio, to whom I am greatly indebted. I thank also the students of my courses for pointing out misprints or unclear points in the preliminary draft of the notes.

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Preface to the Second Edition

The first edition of the book closely reflected the notes of half-semester courses for undergraduate students in mathematics and therefore did not aim to qualify as a monograph on Quantum Mechanics (QM). Rather, it meant to provide a short mathematically minded introduction to the foundations of QM and to its revolutionary philosophical ideas. This explains why many interesting applications and developments were left aside.

The author still thinks that this remains the main motivation for publishing the lecture notes. Nevertheless, the sketchy discussion of the basic mathematical structure in terms of operational arguments might have been profitably expanded for the convenience of the reader, since it hinges on crucial philosophical ideas. For these reasons, Section 1.3 has been considerably improved, by strengthening the operational evidence of a C^* -algebraic structure of the observables, through the realization of their Jordan Banach structure and the Alfsen-Schulz-Størmer theorem.

For the same reasons, a discussion and a comparison with the Dirac-von Neumann principles of Quantum Mechanics is given in Section 3.6, emphasizing the basic distinction, not appreciated in the textbook presentations of QM, between the set of axioms which are direct consequences of the operational definition of measurements and therefore have a general validity for the description of a physical system and those which codify the implications of the Heisenberg uncertainty relations and therefore are peculiar of quantum systems. Actually, for the derivation of the mathematical structure of QM, the operational evidence of complementary observables is shown to be more effective, rather than Heisenberg uncertainty relations.

Another very important and distinctive feature of quantum mechanics versus classical mechanics, not sufficiently emphasized in the standard textbooks and discussed in the added Section 4.4, is that classically dangerous singularities, associated to the unboundedness from below of the potential, becomes harmless in the quantum mechanical noncommutative version. A distinguished example, typically the only one discussed in most textbooks, is the Coulomb bound state problem, but the boundedness from below of the quantum Hamiltonian, i.e. the quantum stability, is actually shared by all the potentials which satisfy the Kato criterium of self-adjointness of the total Hamiltonian.

The second edition differs from the first also for a correction of a few misprints, for the addition of a Bibliography listing the references quoted in the footnotes and for a discussion of coherent states and their relation with classical mechanics.

Contents

0	Introduction	1
1	Mathematical description of a physical system	7
1.1	Atomic physics and the crisis of classical mechanics	7
1.2	Classical Hamiltonian systems	10
1.3	General mathematical description of a physical system . . .	16
1.4	Appendix A : C^* -algebras	26
1.5	Appendix B: Abelian C^* -algebras	28
1.6	Appendix C: Spectra and states	32
2	Mathematical description of a quantum system	39
2.1	Heisenberg uncertainty relations and non-abelianess	39
2.2	States and representations. GNS construction	42
2.3	Gelfand-Naimark theorem: observables as operators	47
2.4	The probabilistic interpretation. * Quantum probability . .	48
2.5	* Quantum logic	54
2.6	Appendix D: States and representations	56
2.7	* Appendix E: Von Neumann algebras	57
3	The quantum particle	59
3.1	The Weyl algebra and the Heisenberg group	59
3.2	Von Neumann uniqueness theorem	62
3.3	The Schrödinger representation and wave function	65
3.4	Gaussian states. Minimal Heisenberg uncertainty	67
3.5	* Coherent states	69
3.6	Remarks on the principles of quantum mechanics	72
3.6.1	Dirac-von Neumann axioms of quantum mechanics .	72
3.6.2	Physical principles of quantum mechanics	75
4	Quantum dynamics. The Schrödinger equation	87
4.1	Quantum dynamics. The quantum Hamiltonian	87
4.2	The dynamics of a free quantum particle	90

4.3	Quantum particle in a potential	92
4.4	Quantum stability	93
4.5	Appendix F: Hamiltonian self-adjointness and dynamics . .	96
5	Examples	105
5.1	Double-slit interference and particle-wave duality	105
5.2	The quantum harmonic oscillator. Energy quantization . .	106
5.3	Quantum particle in a square potential well and in a box .	109
5.4	Quantization of the angular momentum. The spin	111
5.5	The Hydrogen atom	114
5.6	Appendix G: Properties of the Runge-Lenz operator	118
6	* Quantum mechanics and stochastic processes	119
6.1	Quantum mechanics, probability and diffusion	119
6.2	The Feynman path integral	123
6.3	The Feynman-Kac formula	128
6.4	Nelson positivity and uniqueness of the ground state	131
6.5	Quantum mechanics and stochastic processes	133
6.6	Euclidean quantum mechanics	137
6.7	Applications of the functional integral	141
6.7.1	Feynman perturbative expansion	141
6.7.2	Semiclassical limit	143
6.7.3	Ground state properties	147
6.7.4	Coupling constant analyticity	147
6.8	Functional integral and topology	149
6.9	Appendix H: The central limit theorem	159
6.10	Appendix I: Gaussian variables. Wick's theorem	160
6.11	Appendix J: Stochastic processes	162
6.12	Appendix K: Wiener process	165
	Bibliography	169
	Index	177