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Topics in
Number Theory

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Topics in Number Theory

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Preface

Since I left the Institute of Mathematics at Academia Sinica in 1991, I have conducted a graduate course on number theory every year at the National Chung Cheng University. The title of the course is usually “Special Topics in Number Theory” or “Theory of Modular Forms of Several Variables”. Unfortunately, there are no textbooks available for such a course and I had to write my own lecture notes, lifting materials from books by T. Shimura [48], E. Freitag [31], W. L. Baily, Jr. [6] and T. Miyake [42]. To enrich the contents of the lecture notes, I have included some of the results from my own research work.

In such a course as mentioned above, we usually begin with the theory of modular forms of one variable, followed by the theory of zeta functions, new approaches to Bernoulli identities and Kummer Congruences, the theory of modular forms of several variables, the theory of Jacobi forms over Cayley numbers and finally the theory of Jacobi forms of several variables over Cayley numbers. The theory of Jacobi forms is based on my research papers from 1991 to 2000. Students are encouraged to conduct research related to the materials mentioned above. Eventually, a total of 6 Ph.D. students and more than 30 M.Sc. students graduated under this programme.

The theory of modular forms of several variables [31] was initiated by C. L. Siegel around 1945 in an attempt to generalize the theory of modular forms of one variable. The theory was employed by M. Eichler and D. Zagier [15] in 1985 to give a definition of Jacobi forms. Indeed the coefficients of the Fourier-Jacobi expansions of modular forms of rank two are the main sources of Jacobi forms. It is still a difficult problem to construct Jacobi forms besides the Eisenstein series. Even the Eisenstein series of Jacobi

forms of several variables are difficult to construct. All we can say is that many results concerning modular forms of several variables remain to be studied in depth.

Georg-August-Universität Göttingen is the center of development of modular forms with masters like Siegel and U. Christian. I spent a year at Georg-August-Universität Göttingen as a Sonderforschungsbereich (special research fellow) in 1985, three years after my graduation from the University of Chicago in 1982.

It is almost 60 years since the birth of modular forms of several variables in Germany. It is an appropriate time to have a textbook on the theory in English rather than in German. This is my main motivation for writing this book.

Minking Eie
April 19, 2008

Glossary

$\mathrm{SL}_2(\mathbb{R})$	special linear group over $\mathbb{R}$	3
$\mathcal{H}$	upper half-plane of $\mathbb{C}$	3
$\mathrm{SL}_2(\mathbb{Z})$	special linear group over $\mathbb{Z}$	3
$j(g, z)$	$= cz + d$ if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$	5
Γ	$= \mathrm{SL}_2(\mathbb{Z})$	5
$\bar{\Gamma}$	$= \mathrm{SL}_2(\mathbb{Z})/\{\pm E_2\}$, full modular group	5
Γ_0	maximal parabolic subgroup of Γ	5
$\Gamma \backslash \mathcal{H}$	orbit space of Γ acting on $\mathcal{H}$	6
$\mathcal{F}$	a fundamental domain of $\mathcal{H}$ with respect to Γ	6
$G_k(z)$	Eisenstein series of weight k	10
$\mathcal{L}$	$= \{mz + n \mid m, n \in \mathbb{Z}\}$, a lattice of $\mathbb{C}$	10
$\zeta(s)$	Riemann zeta function	11
$\sigma_k(n)$	divisor function	12
B_k	the k th Bernoulli number	12
$E_k(z)$	normalized Eisenstein series of weight k	13
$\Gamma(s)$	gamma function	18
$\pi(x)$	number of prime numbers not exceeding x	18
$\mu(n)$	Möbius function	19
γ	Euler constant	20
$B_n(x)$	the n th Bernoulli polynomial	25
$\zeta(s; \delta)$	Hurwitz zeta function	27
$\mathcal{S}(\mathbb{R}^n)$	space of rapidly decreasing function on $\mathbb{R}^n$	29
$\hat{f}$	Fourier transform of f	29
$\langle \cdot, \cdot \rangle$	inner product	29
$\vartheta(S, z)$	theta series	34

$Z(f, s)$	zeta function attached to a modular form f	36
$T(n)$	Hecke operator	40
R_λ	homothety operator	42
$v_{z_0}(f)$	least integer n such that $b_n \neq 0$	47
$v_\infty(f)$	least integer n such that $a_n \neq 0$	47
$A_k(\Gamma)$	vector space of modular forms of weight k	49
$\Delta(z)$	discriminant	50
$\eta(z)$	Dedekind eta function	50
$\Lambda(f, s)$	$= (2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s+1)$	51
$X(s)$	$= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$	52
$\tau(n)$	Ramanujan's tau function	53
$S_k(\Gamma)$	vector space of cusp forms of weight k	54
$\{\varphi_n(z)\}_{n=1}^N$	an orthonormal basis for $S_k(\Gamma)$	54
$L_k^1(D)$	set of measurable functions on the unit disc	55
$f_\Gamma(z)$	$= \sum_{\gamma \in \Gamma} f(\gamma(z)) j(\gamma, z)^{-k}$	56
$C(\gamma)$	centralizer of γ in $\mathrm{SL}_2(\mathbb{Z})$	58
$\{\gamma\}$	conjugacy class of γ in $\mathrm{SL}_2(\mathbb{Z})$	58
J^ℓ	linear extension of $\prod_{j=1}^\ell X_j^{\alpha_j} \mapsto \prod_{j=1}^\ell \zeta(-\alpha_j)$	67
$\zeta_p(s)$	local Riemann zeta function	78
$S_{p,q}$	double Euler sum	85
$\eta(s; x)$	Lerch zeta function	96
$E_n(x)$	the n th Euler polynomial	96
$\psi(x)$	digamma function	101
$\zeta(\alpha_1, \dots, \alpha_r)$	multiple zeta value or r -fold Euler sum	107
$G_n(p, q)$	Euler sum with two branches	112
$M_n(\mathbb{R})$	matrix ring over real numbers field of size n	131
${}^t X$	transpose of X	131
$\mathcal{H}_n$	generalized (Siegel) upper half-plane	131
p_B	a translation of $\mathcal{H}_n$	133
$\mathrm{GL}_n(\mathbb{R})$	general linear group over $\mathbb{R}$ of size n	133
t_U	a dilation of $\mathcal{H}_n$	133
ι	inversion of $\mathcal{H}_n$	133
$\mathrm{Sp}(n, \mathbb{R})$	real symplectic group of degree n over $\mathbb{R}$	133
$U(n)$	unitary group of size n	139
D_n	generalized disc	140

$M_n(\mathbb{C})$	matrix ring over $\mathbb{C}$ of size n	140
$\mathfrak{H}_n$	Hermitian upper half-plane	142
$\mathbb{H}$	skew field of quaternions over $\mathbb{R}$	142
Ω_n	a tube domain contained in $\mathbb{C}^{2n^2+n}$	142
Γ_n	$= \mathrm{Sp}(n, \mathbb{Z})$	145
Γ_n^0	$= \{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_n \mid C = 0 \}$	145
$\Gamma_n(N)$	principal congruence subgroup of level N of Γ_n	147
Λ_n^*	set of $n \times n$ half-integral symmetric matrices	151
$\sigma(X)$	trace of a matrix X	151
Λ_n	set of $n \times n$ integral symmetric matrices	151
$j(M, Z)$	$= \det(CZ + D)$ if $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(n, \mathbb{R})$	152
$E_k(Z)$	Eisenstein series	152
$\gamma(s)$	$= \prod_{\nu=0}^{n-1} \Gamma(s - \frac{\nu}{2})$	157
$F_k(Z)$	$= \sum_{\det C \neq 0} \det(CZ + D)^{-k}$	158
$\nu(R)$	content of R	158
$S[X]$	$= {}^t X S X$	160
$A(S, T)$	number of solutions to $S[X] = T$	160
$A_q(S, T)$	number of solutions to $S[X] \equiv T \pmod{q}$	160
$d_p(S, T)$	local density at the place p	161
$A_\infty(S, T)$	$= \lim_{N \rightarrow T} \mathrm{vol}(\varphi^{-1}(N)) / \mathrm{vol}(N)$	161
h	number of classes in the genus of a matrix S	162
$\mu(S)$	$= \sum_{i=1}^h w(S_i)^{-1}$	162
$w(S_i)$	order of the group of units of S_i	162
$\vartheta(S, Z)$	theta series on the Siegel upper half-plane	162
(A, B)	trace of AB	162
$A_4(n)$	number of integral solutions to $\sum_{k=1}^4 x_k^2 = n$	166
$(\frac{v}{p})$	Legendre symbol	166
$\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$	Jacobi group	171
$\mathfrak{f}$	a field	174
$\mathcal{C}_{\mathfrak{f}}$	Cayley numbers over $\mathfrak{f}$	174
$\mathcal{O}$	integral Cayley numbers, $\mathbb{Z}$ -module in $\mathcal{C}_{\mathbb{Q}}$	174
$\mathcal{C}_{\mathbb{Q}}$	set of rational Cayley numbers	174
$\mathcal{H} \times \mathcal{C}_{\mathbb{C}}$	product space of $\mathcal{H}$ and $\mathcal{C}_{\mathbb{C}}$	177
$\vartheta_{m,q}(z, w)$	theta series	179
$\Lambda(q)$	$= \{ \lambda + q/m \mid \lambda \in \mathcal{O} \}$	179

$F_q(z)$	$= \sum_{n \geq N(q)/m} \alpha(n, q) e^{2\pi i [n - N(q)/m]z}$	179
$F(z)$	$= {}^t F_q(z)_{q: \mathcal{O}/m\mathcal{O}}$	180
$\Gamma[m]$	principal congruence subgroup of level m of Γ	181
Θ	$= {}^t(\vartheta_{m, q_1}, \vartheta_{m, q_2}, \dots, \vartheta_{m, q_{m^8}})$	182
$E_{k,m}(z, w; q)$	Jacobi-Eisenstein series	184
Γ^∞	$= \{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \}$	184
$E_p(z; q)$	$= \sum_{M: \Gamma/\Gamma^\infty} \overline{\psi_{q,p}(M)} j(M, z)^{4-k}$	184
$E_{k,m}(z; q)$	$= {}^t(E_{q_1}(z; q), \dots, E_{q_{m^8}}(z; q))$	184
Γ_2	subgroup of bi-holomorphic maps of $\mathcal{H}_2$	192
$\mathcal{M}(k, \mathcal{C})$	Maaß space	193
$f_4(Z)$	$= \sum_{h \in \mathcal{O}^2} e^{2\pi i (h^t \bar{h}, Z)}$	193
$J_{k,m}(\Gamma)$	vector space of Jacobi forms	195
$J_{k,m}^0(\Gamma)$	vector space of Jacobi cusp forms	196
$\mathcal{T}_{\mathbb{R}}$	set of 3×3 Hermitian matrices over $\mathcal{C}_{\mathbb{R}}$	199
$X \circ Y$	$= \frac{1}{2}(XY + YX)$	200
$\mathcal{K}$	set of squares $X \circ X$ of elements of $\mathcal{T}_{\mathbb{R}}$	200
$\mathcal{K}^+$	interior of $\mathcal{K}$	200
$\mathcal{H}$	exceptional domain in $\mathbb{C}^{27}$	200
$\mathcal{T}_{\mathcal{O}}$	$= \mathcal{T}_{\mathbb{R}} \cap M_3(\mathcal{O})$	200
$M_3(\mathcal{O})$	set of 3×3 matrices over $\mathcal{O}$	200
$\mathcal{G}$	group of holomorphic automorphisms of $\mathcal{H}$	200
E_7	a type of Lie group	200
Γ'	discrete subgroup of $\mathcal{G}$	200
$E_\ell(Z)$	$= \sum_{\gamma \in \Gamma'/\Gamma'_0} J(\gamma, Z)^\ell$	201
Γ'_0	subgroup of Γ' generated by b_B and t_U	201
$J(\gamma, Z)$	determinant of the Jacobi matrix of γ at Z	201
$j(g, Z)$	a factor of $J(g, Z)$	202
$E_{k,s}(Z)$	$= \sum_{\gamma \in \Gamma'/\Gamma'_0} j(\gamma, Z)^{-k} j(\gamma, Z) ^{-s}$	202
$\vartheta_{m,q}(Z, W)$	$= \sum_{\substack{h = \lambda + q/m \\ \lambda \in \mathcal{O}^2}} e^{2\pi i m [(h^t \bar{h}, Z) + \sigma(h_1, w_2) + \sigma(h_2, w_2)]}$	204
$F(Z)$	$= {}^t(F_{q_1}(Z), \dots, F_{q_{m^{16}}}(Z))$	207
$\Theta(Z, W)$	$= {}^t(\vartheta_{m, q_1}(Z, W), \dots, \vartheta_{m, q_{m^{16}}}(Z, W))$	207
$F_q(Z)$	$= \sum_{\substack{T \in \Lambda_2 \\ T \geq q^t \bar{q}/m}} \alpha(T, q) e^{2\pi i (T - q^t \bar{q}/m, Z)}$	207
Γ_2^0	subgroup of Γ_2 generated by p_B and t_U	208
$E_{k,m}(Z, W; q)$	$= {}^t(E_{q, q_1}(Z), \dots, E_{q, q_{m^{16}}}(Z)) \Theta(Z, W)$	208

$E_{q,p}(Z)$	$= \sum_{M:\Gamma_2/\Gamma_2^0} j(M, Z)^{4-k} \overline{\psi_{q,p}(M)}$	208
$E_4(Z)$	modular form of weight 4 on $\mathcal{H}$	209
$E(Z)$	$= f_4(Z) + \sum_{m=1}^{\infty} \varphi_m(Z_1, W) e^{2\pi i m z_3}$	210
$E^2(Z)$	$= \sum_{T \in \mathcal{T}_{\mathcal{O}} \cap \mathcal{K}^+} b(T) e^{2\pi i(T, Z)}$	211
$\Gamma_2 \setminus \mathcal{H}_2$	a fundamental domain of $\mathcal{H}_2$ for Γ_2	213
$dXdY$	Euclidean measure on $\mathcal{H}_2$	213
$\nu_p(m)$	p -adic valuation	216
$J_{k,m}^0(\Gamma_2)$	vector space of Jacobi cusp form	216
$\mathbb{Z}_p$	ring of p -adic integers	219
$\mathbb{Q}_p$	field of p -adic numbers	226
$\mu(x)$	p -adic measure	233

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