Basic Control Volume Finite Element Methods for Fluids and Solids

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Basic Control Volume Finite Element Methods for Fluids and Solids

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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

**BASIC CONTROL VOLUME FINITE ELEMENT METHODS FOR FLUIDS
AND SOLIDS**

IISc Research Monographs Series — Vol. 1

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ISBN-13 978-981-283-498-0

ISBN-10 981-283-498-2

Printed in Singapore.

Series Preface

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To my mentors, advisors, colleagues, and students

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Preface

The advent of the digital computer in the middle of the last century initiated a rapid and continued growth in the development of computational tools for solving field problems. In the early days of these developments, researchers worked on a relatively small set of methods, applying them to a broad range of mechanics problems from stress and strain in solids through to fluid flow. As the field matured, however, distinct camps of researches based around methods and problems were formed. In extremely broad terms, the development of methods were split between those based on finite difference approaches and those based on finite element approaches; likewise applications were split between solids and fluids.

As the computational modeling field moved forward, other classes of solution methods were developed. Of particular note were control volume/finite volume methods. An immediate appeal of such methods was their obvious connection, through explicit discrete balance equations, to the physics of the problem at hand. Early control volume developments used finite difference methods to arrive at appropriate discrete equations. It was rapidly realized, however, that control volume solutions could also be constructed through the use of finite element technologies. Thus, control volume methods are viewed by some researchers as bridging between finite difference and finite element methods, with the ability to adopt and adapt the advantage of these methods while neglecting the drawbacks.

The Control volume methods that seem to obtain the maximum advantage of this hybrid view point are those based on finite element

technologies, referred to as Control Volume Finite Element Methods (CVFEM). A notable feature of this class is the relative ease by which they can be applied to both solids and fluids problems. As such, the current research focused on solving multi-physics problems has spurred a significant interest in developing CVFEM solutions.

The central aim of this monograph is to introduce the basic and essential ingredients in control volume finite element methods. It is felt that this introduction will provide researchers with the critical background and base tools that will allow for more general application of CVFEM. Further, looking toward future multi-physics applications and trying to recapture the more comprehensive approach of the early days of computational mechanics, this monograph develops the basic constructions of CVFEM in the context of solving fundamental problems in both solids and fluids. This approach serves to fully highlight the generality and flexibility of CVFEM.

As with all efforts of this nature there is a host of people to thank. In general terms, I would first like to thank all of my mentors, advisors, colleagues, and students who have greatly contributed to my current understanding of computational mechanics. More explicit thanks need to go to the faculty of the Department of Mechanical Engineering at the Indian Institute of Science in Bangalore for providing motivation and support in this effort. I am also indebted to Mr Jim Hambleton who acted as a great sound board during the writing and provided a proof reading of the draft text.

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