Classical Mechanics and Relativity

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Classical Mechanics and Relativity

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Preface

Classical Mechanics is the most basic subject of any degree course in physics, and the subject is now so well established that its main building blocks, the Lagrangian and Hamiltonian methods, provide it with a universal character which forms the basis of other fundamental areas of physics, such as electrodynamics, quantum mechanics and statistical physics. A thorough understanding of classical mechanics is therefore a far-reaching help for the understanding also of these other areas. Classical mechanics became a theoretical subject with Newton's laws of motion, and attained its universality with the replacement of these laws by Hamilton's action principle as the basic axiom. This principle is of such fundamentality that today almost any model theory proposed in particle physics is first formulated in terms of such an action.

It is common knowledge today that Einstein's theory of relativity, in both its Special and General forms, extended Newton's mechanics to spectacularly precise confirmations by experiment, the Special Theory, for instance, in electrodynamics, and the General Theory in celestial mechanics. Einstein's mechanics is widely regarded as unusually complex and is popularly considered difficult to grasp. The theory of relativity is therefore rarely presented along with ordinary classical mechanics. In some sense this is a pity since the student of classical mechanics — and the latter includes always Newton's gravitational law and planetary motion — knows very well that there is more to it, that Einstein's relativity provides corrections, but with no reference to this he is supported in his belief that this is a theory beyond his understanding. This does not have to be the case, and in fact can be expected to change, since *e.g.* texts on space science assume already a first acquaintance with general relativity. Therefore we make an attempt here to direct the reader's view from as early a stage as possible to the ultimate theory of Einstein, so that even if the reader's prime interest is focussed on classical mechanics, he can turn the pages and explore the steps to this ultimate theory. Thus, an example in Lagrangian dynamics serves to derive the full Kepler problem from the Schwarzschild metric; the Kepler problem is then treated in full

in the context of central forces, and later Newton's gravitational force and the Schwarzschild metric are obtained in Einstein's theory of relativity. An essential complementary aspect is the stepwise generalization of rotations to spacetime and general coordinate transformations, much of which can be grasped by analogy.

Apart from the inclusion of an introduction to relativity, the text covers most of the topics usually treated in standard texts on classical mechanics, such as for instance, the very wellknown book of H. Goldstein, and as in this and in texts of other authors, such as T. W. B. Kibble and F. H. Berkshire, frequent reference is made to electrodynamics, especially in examples. Our presentation here is in some respects more detailed and includes a larger number of worked problems, including a number of problems given as unsolved in the text of H. Goldstein. As in the author's earlier texts on electrodynamics and quantum mechanics, phrases like "it is easy to see" or "it is easy to show" are consistently avoided since if true the explanations can readily be provided by an author, and a condensed form which leaves many derivations to the reader seems inappropriate for the companion text of a basic course. In addition, for the benefit of readers who are new to the field, some vital points are occasionally repeated or re-emphasized. Thus, as a rule, besides detailed explanations, all derivations are given, assuming that it is easier for a reader to skip or run through given calculations than to reproduce these himself at numerous points of the text, which in any case leaves the reader a lot of other points to struggle with for himself.* A proper understanding requires a thorough understanding of each and every step, a mental struggle being in any case unavoidable. As the philosopher A. Schopenhauer remarked: The process of cognition is the path from a paradox to triviality. Once something has been thoroughly understood, it appears trivial.

Harald J. W. Müller-Kirsten

*Throughout the text names are cited without first names, *e.g.* Einstein, and only these are named in the index. Names in footnotes, usually with reference to literature, are given with initials of first names, but are not cited in the index.