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INTRODUCTION TO THE THEORY OF WEIGHTED POLYNOMIAL APPROXIMATION

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World Scientific

Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Mhaskar, H. N. (Hrushikesh Narhar), 1956-

Introduction to the theory of weighted polynomial approximation /
by H. N. Mhaskar.

p. cm. -- (Series in approximations and decompositions : vol. 7)

Includes bibliographical references (pp. 355-379) and index.

ISBN 9810213123 (Singapore : alk. paper)

1. Approximation theory. 2. Orthogonal polynomials. I. Title.

II. Series: Series in approximations and decompositions ; Vol. 7.

QA221.M52 1996

511'.42--dc20

96-35757

CIP

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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Printed in Singapore by Uto-Print

Approximations and Decompositions

During the past decade, Approximation Theory has reached out to encompass the approximation-theoretic and computational aspects of several exciting areas in applied mathematics such as wavelets, fractals, neural networks, and computer-aided-geometric design, as well as the modern mathematical development in science and technology. The objective of this book series is to capture this exciting development in the form of monographs, lecture notes, reprint volumes, text books, edited review volumes, and conference proceedings.

This seventh volume is a monograph on the approximation of functions on the whole real line. Written by an expert in approximation theory, this book differs from the existing texts and survey articles on this and related topics in that the subject is presented from the point of view of approximation theory, rather than that of potential theory. Hence, the basic topics of approximation on a bounded interval, such as interpolation and quadrature, Favard-type estimates, as well as the K-functional and degree of approximation are first discussed, and the main theme of the monograph is to explore various interesting and nontrivial generalizations to the theory of approximation on the real line. The series editor would like to congratulate and thank the author, Professor H. N. Mhaskar for this excellent and innovative contribution.

World Scientific Series in

APPROXIMATIONS AND DECOMPOSITIONS

Editor-in-Chief: CHARLES K. CHUI

Texas A&M University, College Station, Texas

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To
My Mother

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Preface

The subject of approximation of functions by trigonometric and algebraic polynomials is a very classical one. Besides being routinely used in many fields of sciences and engineering, it provides a "role model" for the study of other approximation processes, such as approximation by splines, rational functions, exponential sums, neural networks, and wavelets. However, if one is interested in approximating functions on the whole real line by unbounded functions such as polynomials, then one is forced to study the problem in weighted function spaces.

The classical Bernstein approximation problem seeks conditions on the weight functions w such that the set of functions $\{w(x)x^n\}_{n=0}^{\infty}$ is fundamental in the class of continuous functions on $\mathbb{R}$, vanishing at infinity. Many people worked on this problem for at least 40 years. In 1970's G. Freud started a program to develop a richer theory for weighted polynomial approximation, which is analogous to the known classical theory of trigonometric approximation on the circle. The plan was to start with well studied weight functions, such as $\exp(-x^2/2)$, and then generalize the weight functions to which the theory can be applied. Necessarily, the beauty of the subject is in that many new techniques have to be introduced, rather than simply the final results themselves.

In this book, we have attempted to explain a variety of different techniques and ideas which have contributed to this subject in its course of successive refinements during the last 25 years. There are other books and surveys reviewing the ideas from the perspective of either potential theory or orthogonal polynomials. The main thrust of this book is to introduce the subject from an approximation theory point of view. Thus, the main motivation is to study analogues of results from classical trigonometric approximation theory, introducing other ideas as needed. It is not our objective to survey the most recent results, but merely to introduce to the readers the thought processes and ideas

as they are developed. Some results in the book are new, and we have often pursued some of the old results from a new perspective, seeking to clarify the plethora of conditions under which the results appear in journal articles.

In Chapter 1, we review some elementary facts from the theory of orthogonal polynomials, which are used throughout the book. Next, we start with an exposition on the question of degree of approximation; *i.e.*, the connection between the accuracy of approximation and the constructive properties of the “target function”. Partly because of the absence of a convolution structure, the theory is extremely elaborate. In Chapter 2, we study polynomial approximation on a compact interval in order to illustrate the basic ideas to be explored in Chapters 4 and 5. Chapter 3 develops many technical estimates regarding the “Freud polynomials”, which are essential in these chapters. We have also presented the theory in a way so that a lot can be achieved with only rudimentary analysis, and yet a general theory will be completed when certain basic inequalities are proved using deeper ideas in Chapter 6. The question of degree of approximation is studied in Chapter 4. Chapter 5 is devoted to the evaluation of certain K -functionals arising in this theory. For the convenience of the reader who may wish to skip the details of Chapter 3, the relevant facts are summarized in these two chapters as needed. Chapter 6 is one of the central chapters of the book, where many important technical results are proved. Although the results were first developed using the language of potential theory, we have shown how little potential theory was actually needed; our discussion is based on certain well known facts in the theory of Fourier series. These facts are reviewed both as needed, and in the appendix. Chapter 7 deals with the problem of approximation of entire functions. The theory of weighted approximation surpasses the classical theory here: one can obtain a precise expression for the type and order of the target function in terms of the degree of weighted approximation. Chapter 8 contains further technical results regarding Freud polynomials. In Chapter 9, we apply these results to the study of orthogonal polynomial expansions, polynomials of Lagrange interpolation, and quadrature processes. Chapters 10 and 11 are of a more advanced nature. In Chapter 10, we utilize most of the results in Chapters 3, 6, and 8 to study the closure of certain “weighted polynomials”, and the asymptotic behavior of the leading coefficients of the Freud polynomials. In Chapter 11, we study miscellaneous subjects, which could not be integrated into the rest of the book. Thus, we briefly present the potential theory ideas, and their application to incomplete polynomials and related topics. We also apply the theory to the complexity problem in the theory of neural networks, or more precisely, Gaussian networks. We make some remarks about the con-

struction of "wavelets" from an arbitrary system of orthogonal polynomials. The Appendix reviews certain facts which are sometimes not covered in first courses on mathematical analysis, but which are used often in the book. This includes also a brief review of the theorems from trigonometric approximation, which provide a "role model" for the results in the book. We have included a nonexhaustive bibliography of the subject, which is expected to provide the reader a starting point for further research. For the convenience of typing, we have made a separate section called "References", which only lists those papers and books, which are actually cited in the notes.

This book is intended to be self-contained, although the reader is expected to be familiar with rudimentary real and complex analysis. It will also help to have studied elementary trigonometric approximation theory, and have some exposure to orthogonal polynomials. As a first reading, those who are primarily interested in approximation theory should probably read Chapters 1, 2, 4, 5, 7, 9, and Sections 11.2, 11.3; trusting the results in the other chapters. Those who are primarily interested in the theory of orthogonal polynomials will hopefully enjoy Chapters 1, 3, 6, 8, 9, 10, and Sections 11.2, 11.3. Readers who are primarily interested in potential theory are likely to find it interesting to read Chapter 6, Chapter 10, and Sections 7.2, 8.3, 9.3, and 11.1. The upcoming book of Saff and Totik contains more on these subjects.

In learning this subject, we have benefitted from the discussions and collaborations with many mathematicians. It is impossible to name all of these, but we take this opportunity to express our appreciation to all of them. In the context of this book, we are particularly grateful to Professors R. Al-Jarrah, H. -P. Blatt, R. Bojanic, P. L. Butzer, E. W. Cheney, C. K. Chui, Z. Ditzian, P. Erdős, M. v. Golitscheck, R. Grothmann, T. Kilgore, G. G. Lorentz, D. S. Lubinsky, C. A. Micchelli, P. Nevai, J. Prestin, E. B. Saff, J. Szabados, and V. Totik for their encouragement and comments on various parts of the book, to Mrs. M. Chui and Professor J. Stöckler for their technical assistance, and to the publisher for their patient encouragements and proddings. We are also grateful to Dr. Grothmann of Katholische Universität, Eichstätt for his careful reading and comments on the entire book, as well as to Mr. N. Hahm, currently at University of Texas at Austin, for having tirelessly proof-read the manuscript. We thankfully acknowledge support from the Alexander von Humboldt Foundation, the National Science Foundation, the Air Force Office of Scientific Research, and the California State University, Los Angeles during the preparation of this book.

Los Angeles, May, 1996

H. N. Mhaskar

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