THE LANGEVIN EQUATION

**With Applications in Physics,
Chemistry and Electrical Engineering**

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THE LANGEVIN EQUATION

With Applications in Physics,
Chemistry and Electrical Engineering

W T Coffey

Trinity College, Dublin

Yu P Kalmykov

Russian Academy of Sciences, Moscow

J T Waldron

Dublin City University, Dublin

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Dedicated to the memory of our dear friends

Herbert Fröhlich

and

Hannes Risken

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PREFACE

This book may be said to have its origins in a remarkable conference on Brownian motion held at the School of Theoretical Physics, Dublin Institute for Advanced Studies in August 1976 during the course of which several of the problems we shall discuss were first posed.

One of our main objectives in writing the book is to demonstrate how the concept of the equation of motion of a Brownian particle - the Langevin equation - first formulated by Langevin in 1908 - so making him the founder of the subject of stochastic differential equations, may be radically extended to solve the nonlinear problems which arise in the theory of the Brownian motion in a potential. This approach to the subject enables one to completely dispense with the underlying probability density diffusion equation - the Fokker-Planck equation, with all its attendant mathematical complications. These are particularly pronounced in the theory of rotational Brownian motion on which heavy emphasis is laid throughout the book.

The basis of our treatment is to regard the Langevin equation for a set of random variables (the time average of which is the desired quantity) starting from a set of sharp values of them at a given time as an *integral equation* for their values at a later time. The time average of this equation, calculated in accordance with the Stratonovich rule (as is also used to calculate the drift coefficient in the corresponding Fokker-Planck equation) may then be expressed in terms of a *deterministic* equation of motion for the set of sharp starting values. The method may be applied repeatedly to generate the hierarchy of differential-recurrence relations governing the time evolution of the averages usually generated by the Fokker-Planck equation. Hence that equation is shown to be redundant and any desired average property of a system governed by a Langevin equation may be *directly* calculated from that equation so extending Langevin's treatment of the Brownian motion to nonlinear systems. The problem of calculating the averages ultimately reducing to the task of diagonalising the system matrix, just as in the state space approach to dynamical systems theory as the differential-recurrence relations may always be written as a first order linear matrix differential equation with constant coefficients. This representation also has the outstanding advantage that it avoids the difficulties one encounters (with special functions etc.) by regarding the calculation of the eigenvalues of the Fokker-Planck equation as a Sturm-Liouville problem. All that is required are simple properties of matrices. The matrix representation of the solution also yields a simple general formula for the correlation time (in linear response theory the area under the curve of the normalised decay function) in terms of the inverse of the system matrix and the initial value vector with that vector being in turn a function of *equilibrium averages* only.

Another important result which follows from our treatment is that on extending the Cresser-Risken algorithm (Chapter 6) for the solution of three term recurrence-relations it is possible to have exact solutions for the Laplace transform of the after-effect function, the correlation time (which appears in terms of special functions) etc. for a variety of relaxation problems. The representation of the exact solution for the correlation time in terms of sums of products of special functions (usually confluent hypergeometric (or Kummer) functions) has the advantage that it allows that quantity to be rendered in integral form, the method of steepest descents may then be easily applied to determine the asymptotic behaviour.

In addition, for relaxation problems involving a strong external field and a bistable potential, (see Chapter 8) we demonstrate that the conventional assumption that a single

decay mode dominates no longer invariably holds. This behaviour occurs for external fields less than the critical field at which the two minima structure of the potential disappears. This in turn means that the assumption that the correlation time may be closely approximated by the inverse of the smallest non-vanishing eigenvalue (for sufficiently high barriers effectively the inverse of the Kramers escape rate) is no longer absolutely true. Such behaviour arises because only in certain cases such as relatively weak external fields can the contribution of the other decay modes be neglected.

Our ideas owe much to the work of the late Hannes Risken and the important suggestion of a change of variable in the Langevin equation to the quantity the average value of which one wishes to calculate, made by Frood and Lal in 1975, the other central idea being Doob's interpretation in 1942 of the Langevin equation as an integral equation (the relevant papers and books are cited in the text). This is treated in detail in Chapters 5 - 8.

The book is arranged in a manner such that much of the material may be accessible to a beginning graduate or advanced undergraduate student. A synergetic approach in the spirit of Haken and the principles laid down by the Editors of the series *Advances in Chemical Physics* being followed as far as is possible with examples being taken from fields as diverse as laser physics and paleomagnetism. In particular, large parts of the material presented in Chapters 1 and 3 have been given as a final year undergraduate course in the School of Engineering, Trinity College Dublin. Chapter 1 being designed *inter alia* to serve as an expanded guide to the classical literature e.g. the Dover publication *Selected Papers on Noise and Stochastic Processes*, edited by Nelson Wax. Chapter 2 is devoted to the detailed development of our Langevin method for the calculation of averages described above. Chapters 4 to 10 on the other hand are concerned with particular applications of the theory, to dielectric relaxation, superparamagnetism, ring laser gyros to name but a few.

There are a number of organisations and individuals who have greatly helped us which we would like to thank, in particular, Derrick Crothers, James McConnell, Brendan Scaife, (with whom the tradition of dielectric studies at Trinity College Dublin and the Dublin Institute for Advanced Studies began) Myron Evans (Series Editor) James Calderwood, Roy Chantrell, Francis Farley, Akio Morita, Jagdish Vij, Peter Corcoran, Estomih Massawe, Paul Fannin, Lawrence Geoghegan, Patrick Cregg, Kevin Quinn, Vladimir Gaiduk, Jean Louis Dormann, Jean Louis Déjardin, Stuart Rice, Tony Wickstead, Wolfram Schröer, Maxi San Miguel and Eugene Walsh. We would also like to thank the Trinity College Dublin Trust, the Soros Foundation, the Royal Irish Academy, the Royal Society, the Nuffield Foundation, the British Council and Forbairt for travel grants etc. We thank our families who bore with us during the heavy work of writing the volume and last but not least Ms. Brenda McDonald for her magnificent typing efforts.

Trinity College Dublin
27 July 1995

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