GENERALIZATIONS OF STEINBERG GROUPS

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**SERIES IN
ALGEBRA
VOLUME 4**

GENERALIZATIONS OF STEINBERG GROUPS

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Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Fournelle, T. A. (Thomas A.)

Generalizations of Steinberg groups / T.A. Fournelle, K.W. Weston.

p. cm. -- (Series in algebra ; v. 4)

Includes bibliographical references (p. 225-226) and index.

ISBN 9810220286 (alk. paper)

1. Group theory -- Relations. I. Weston, K. W. (Kenneth W.)

II. Title. III. Series: Series in algebra ; vol. 4.

QA174.2.F68 1996

512'.55--dc20

96-29046

CIP

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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Printed in Singapore by Uto-Print

For

KATIE AND ISABELLE

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Preface

This book arose out of the joint work of the authors over the last six years. Some others have joined in this work at various times. Of these, Said Sidki has invested the greatest amount of effort. The direct origin of the book is a 1968 paper by the second author. A more recent paper in 1985 reawakened his interest in the subject. The first author became interested after this and with the help of the group theoretic programming language CAYLEY was able to shed some light on what groups admit verbal embeddings. While verbal embeddings at first seemed to be rather rare, computer calculations suggested that they were rather common. Moreover, the groups that appeared seemed very interesting. At this point the concept of verbal embedding was transformed into the concept of a linkage group and this was considered a method of constructing groups.

In order to describe further the groups that arise as linkage groups the authors were naturally led to the concept of a linkage Lie algebra and its associated Lie group. This in turn became to be considered as a method of constructing (usually infinite dimensional) Lie algebras.

This book describes all of the above concepts and many of the computer calculations and algorithms that led to these concepts. Some of the proofs may be a bit longer than they have to be. However, it seemed important in some sections to lengthen a proof to add clarity. The notation is generally standard and follows that of the books of D.J.S. Robinson which are listed in the bibliography. Note that functions and operators act on the right. Although some people feel uncomfortable with this, it seems natural to at least one of the authors.

Although the main intent of this work is to construct certain types of groups from a combinatorial point of view, it touches on many algebraic topics. These topics include finite simple groups, classical Lie algebras, infinite dimensional Lie algebras, the combinatorial theory of groups and algebras, Steinberg groups, and logic. However, it deals with these topics in a non-standard manner. Thus, this work is somewhat experimental. It is hoped that it will be found interesting and useful.

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