Collected Papers
— of —
V. K. Patodi

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editors

M. F. Atiyah

University of Cambridge, UK

M. S. Narasimhan

International Centre for Theoretical Physics

Trieste, Italy

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Foreword

Vijay Kumar Patodi was born on March 12, 1945, in Guna, a town in central India. After early education at Vikram University, Ujjain (B.Sc.) and Banaras Hindu University (M.Sc.) and a stay of one year at the Centre for Advanced Study in Mathematics, Bombay University, he joined — in 1967 — the School of Mathematics of the Tata Institute of Fundamental Research, Bombay, as a Research Scholar.

Patodi's first work was a proof of a conjecture of McKean and Singer. An analytical approach via the heat equation yields easily a formula for the index of a linear elliptic operator on a compact manifold; however, the formula involves an integrand containing too many derivatives of the symbol of the operator, while from the Atiyah–Singer index theorem one would expect only second derivatives to figure. In the case of the operator $d + d^*$ on a compact Riemannian manifold, whose index is the usual Euler–Poincaré characteristic, McKean and Singer conjectured that this integrand would coincide with the integrand in the Gauss–Bonnet–Chern formula. However, computational difficulties in proving such a result were formidable. Patodi produced a proof that such a fantastic cancellation of higher derivatives did indeed take place. He was working with local coordinates and the algebraic manipulations were difficult to understand. The experts to whom the proof was sent, while expressing interest, were not completely convinced. Patodi gave some private talks to me and Ramanan explaining the proof. Ramanan and I were able to cast some of the calculations in a global form, and formulated and proved some “super” algebraic lemmas, which appear in section 2 of his paper [1]. This considerably simplified the proof and the paper was published in the *Journal of Differential Geometry* in 1971.

He then attacked the much more complicated and difficult problem of proving the Hirzebruch–Riemann–Roch theorem for complex manifolds by using the heat equation. However, the expected cancellations did not seem to occur, and I suggested to him that he should look at the case of a Kähler manifold. Within a week he came up with a proof in the Kähler case — which was a tour de force — and also with a proof of the Hirzebruch index theorem. These results immediately attracted the attention of the mathematical community and his reputation as an outstanding mathematician was established.

He obtained his Ph.D. from Bombay University in 1971 for his dissertation entitled “Heat equation and the index of elliptic operators”, written under the guidance of S. Ramanan and myself.

At this stage I felt that it would be highly profitable for Patodi to come into contact with Atiyah, and to work with him. Atiyah was then (1971) a permanent member of the Institute for Advanced Study, Princeton. With the help of A. Borel,

a visit for Patodi to the Institute for Advanced Study was arranged. His stay in U.S.A. resulted in his celebrated joint work with Atiyah, Bott and Singer.

On his return to India from U.S.A. he continued to produce work of high quality in spite of his failing health. He took an active part in the working of the Mathematics Faculty of the Tata Institute.

Patodi became an Associate Professor at the Tata Institute in 1973, and a Full Professor in 1975. He gave an invited address at the International Congress of Mathematicians held in Vancouver in 1974.

Patodi was a mathematician of real originality and technical power. He had a charming and quietly confident personality.

His mathematical achievements are all the more remarkable because of the handicap of his frail health. His physical condition deteriorated and led to renal failure in 1975. In spite of intensive medical care, he passed away on 21 December 1976, at the age of 31.

M. S. Narasimhan

I got to know Patodi during the two years (1971–73) that he spent at the Institute for Advanced Study. During this time we collaborated very closely, working at different times with Bott and Singer. I came to appreciate Patodi's remarkable mathematical talents and his charming and modest personality. He had a quiet but engaging sense of humour and a simple life style which survived the transition from India to Princeton. He took some pride in entertaining his close friends to the delights of an elaborate vegetarian cuisine. Sadly he had serious health problems which were not always attended to with sufficient care. As with his more famous predecessor Srinivasa Ramanujan, these medical complications led to his untimely death, just as his mathematical reputation had been firmly established.

When Patodi first came to Princeton there were two new proofs of the index theorem in the context of Riemannian Geometry. One, due to Patodi himself, was a direct and extremely skilful direct calculation (as explained by Narasimhan). The other was the *a priori* proof by Gilkey which asserted that higher derivatives had to disappear. Unfortunately Gilkey's proof was extremely algebraic, with no real use being made of differential geometry and the tensor calculus. In comparing these two different proofs and trying to understand them, Bott and I, with Patodi's help, finally ended up with a third proof [4] which tried to get the best of both worlds.

Interestingly enough, the interaction with theoretical physics of the past two decades, has produced yet more variants of the proof and these are closer to Patodi's

original direct proof. “Super-symmetry” appears to lie behind the sophisticated algebraic identities of Patodi’s approach.

These new proofs of the index theorem, being essentially “local”, paved the way for the joint papers (with Singer) on the case of manifolds with boundary.

When Patodi came to Princeton, I was already struggling to formulate some version of the Hirzebruch signature theorem for manifolds with boundary which would include various motivating special cases. Once the problem had reached the relevant stage, Patodi simply retired for a few days to carry out the kind of detailed calculation that was his forte. He returned in triumph to announce that he had verified the desired conjecture. Thus encouraged, Singer and I then had to grapple with understanding Patodi’s calculations in our own terms, so that we could formulate and prove the appropriate generalizations. It was a great collaboration, exploiting the different talents of the participants, and I am glad it came to a successful conclusion. Tragically Patodi had died before all these results appeared in print and, as his publications show, several further collaborations with others appeared posthumously.

Michael Atiyah

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Contents

Foreword	v
<i>by M. S. Narasimhan and M. Atiyah</i>	
[1] Curvature and the eigenforms of the Laplace operator	1
<i>J. Differential Geometry</i> 5 (1971), 233–249.	
[2] An analytic proof of Riemann–Roch–Hirzebruch theorem for Kaehler manifolds	18
<i>J. Differential Geometry</i> 5 (1971), 251–283.	
[3] Curvature and the fundamental solution of the heat operator	51
<i>J. Indian Math. Soc.</i> 34 (1970), 269–285.	
[4] (with M. F. Atiyah and R. Bott) On the heat equation and the index theorem	68
<i>Invent. Math.</i> 19 (1973), 279–330.	
Errata, <i>loc. cit.</i> 28 (1975), 277–280.	
[5] Holomorphic Lefschetz fixed point formula	124
<i>Bull. Amer. Math. Soc.</i> 79 (1973), 825–828.	
[6] (with M. F. Atiyah and I. M. Singer) Spectral asymmetry and Riemannian geometry	128
<i>Bull. London Math. Soc.</i> 5 (1973), 229–234.	
[7] (with M. F. Atiyah and I. M. Singer) Spectral asymmetry and Riemannian geometry, I, II, III	134
<i>Mathematical Proc. Cambridge Philosophical Soc.</i> 77 (1975), 43–69, 405–432; 77 (1976), 71–99.	
[8] Riemannian structures and triangulations of manifolds	218
<i>Proc. International Congress of Mathematicians</i> , Vancouver, 1974.	
[9] A combinatorial formula for Pontrjagin classes	223
Istituto Nazionale di Alta Matematica, <i>Symposia Matematica</i> 20 (1976), 497–505.	

- [10] (with J. Dodziuk) Riemannian structures and triangulations of manifolds 232
J. Indian Math. Soc. **40** (1976), 1–52.
- [11] (with H. Donnelly) Spectrum and the fixed point sets of isometries II 284
Topology **16** (1977), 1–11.