Riemannian Manifolds of Conullity Two

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Introduction

In several fields of science, and in particular in differential geometry, Riemannian symmetric spaces play a prominent role. The analytic condition expressing (local) symmetry, $DR = 0$, implies the local homogeneity and hence, the Riemann curvature tensor R of a (locally) symmetric space is the “same” at each point. Moreover, R is not arbitrary: E. Cartan proved that an algebraic curvature tensor K belongs to some symmetric space if and only if it satisfies the additional condition $K(X, Y) \cdot K = 0$. It is then a natural step to define a (broader) class of Riemannian manifolds by requiring that they satisfy the curvature condition $R(X, Y) \cdot R = 0$. Riemannian manifolds satisfying this condition are called *semi-symmetric spaces*. Actually, the above condition means that, at every point p of a semi-symmetric space (M, g) , the Riemann curvature tensor R_p belongs to some symmetric space, but this symmetric space can change from point to point. For example, it is easy to see that all two-dimensional Riemannian manifolds are semi-symmetric.

The first mathematician who paid attention to the class of semi-symmetric spaces was E. Cartan, in connection with his fundamental research on (locally) symmetric spaces. Many other mathematicians continued this study (see Chapter 1). A major and fruitful impulse was provided by K. Nomizu in 1968: in [98], he conjectures that all complete, irreducible semi-symmetric manifolds of dimension greater than or equal to three are locally symmetric. His conjecture was refuted in 1972 by H. Takagi ([141]) who constructed a hypersurface in $\mathbb{R}^4$ which provided the first counterexample, and by K. Sekigawa ([119]) who published counterexamples of arbitrary dimensions. These

results raised the problem of classifying all semi-symmetric spaces.

Between 1982 and 1985, Z. I. Szabó published three fundamental papers on the subject: in [138], he gives the full local classification of Riemannian semi-symmetric spaces; [139] treats complete semi-symmetric spaces and [140] deals with the classification and construction of complete hypersurfaces of Euclidean spaces which are semi-symmetric with respect to the induced metric.

The local structure theorem in [138] (see Theorem 2.9) says that every locally irreducible semi-symmetric space falls into one of the following three classes (the terminology is due to [59]) :

- 1) the “trivial” class: all locally symmetric spaces and all two-dimensional Riemannian surfaces;
- 2) the “exceptional” class: elliptic cones, hyperbolic cones, Euclidean cones and Kählerian cones;
- 3) the “typical” class: Riemannian spaces foliated by Euclidean leaves of codimension two.

Whereas the “trivial” semi-symmetric manifolds are well-understood and the “exceptional” ones are described and constructed explicitly in [138] and [139], the foliated semi-symmetric spaces are given here only *implicitly*, i.e., Szabó does not provide explicit expressions for the metric of such spaces, but he only gives an integrable system of non-linear partial differential equations and the exact number of solutions: the solutions in dimension n depend in general on $(n-2)(n+3)/2+4$ arbitrary functions of two variables and $(n-2)(n+3)/2$ arbitrary functions of one variable. In [138], however, no explicit solutions are given. As far as we know, the first explicit examples depending *essentially* on one arbitrary function of *two* variables were constructed by F. Tricerri, the second and the third author in [73], [74] as generalisations of two examples of K. Sekigawa ([124]).

In 1991, the second author made a systematic study of three-dimensional foliated semi-symmetric spaces ([59]). He deduces a system of partial differential equations equivalent to that of Szabó, but by a different method and in a different form. Then he introduces the geometric notion of an “asymptotic foliation” (whose leaves are of

codimension one) and divides the class of semi-symmetric manifolds into four subclasses: the planar, the hyperbolic, the parabolic and the elliptic spaces, depending on the number of asymptotic foliations they admit. Further, he succeeds in solving the basic system of partial differential equations to obtain *explicit* expressions for the metrics of such spaces, except for the elliptic case, where only a quasi-explicit formula is given. Finally, he proves that the local isometry classes of these metrics can be parametrised by three arbitrary functions of two variables, which is much less than the number of arbitrary functions derived formally by Szabó (his formula gives seven functions of two variables for $n = 3$).

In this book, we will study foliated semi-symmetric spaces (in fact, *Riemannian manifolds of conullity two*) of arbitrary dimensions. We derive the basic system of partial differential equations and define “asymptotic foliations” also in higher dimensions. Moreover, we introduce a new geometric invariant for a foliated semi-symmetric space, the “algebraic rank”. Using these two concepts, we give various classifications and examples. We apply our results to the theory of hypersurfaces with type number two in the Euclidean space $\mathbb{R}^4$. Then we extend, in dimension three, our results to a broader class of Riemannian manifolds which are “elliptic” and “hyperbolic” analogues of spaces of conullity two. Finally, we add a special *expository* (and up-to-date) survey on curvature homogeneous spaces. This last chapter is related to the previous topic rather by the similarity of *methods* than by contents.

The contents of the subsequent chapters is as follows. In Chapter 1, we give a rigorous definition for semi-symmetric spaces and we inform briefly about the extrinsic analogue of these spaces (called semi-parallel submanifolds). In the last section of this chapter, we give a concise overview of the early research about semi-symmetric spaces. This survey is certainly not exhaustive. We refer to the quoted papers for more detailed information and further references.

Chapter 2 is devoted to the fundamental local structure theorem by Z. I. Szabó. We indicate the major steps in his reasoning and take a closer look at the different irreducible factors in which a general semi-symmetric space can be decomposed (locally).

In Chapter 3, we start the (local) investigation of foliated semi-symmetric spaces of arbitrary dimension. Generalising the procedure known for the three-dimensional case, we first derive the basic system of partial differential equations for our problem. Next, we find some first integrals for this system and present two simplifications of the problem, which will both be useful in what follows: the first one already in Chapter 4, the second one throughout Chapter 5 to Chapter 9.

Chapter 4 deals with a special subclass of semi-symmetric spaces, namely those for which the curvature tensor is the same at each point, the so-called *curvature homogeneous semi-symmetric spaces*. Using the local structure theorem by Szabó, we reduce the classification problem to the study of foliated semi-symmetric spaces with constant scalar curvature. Using the techniques developed in Chapter 3, we determine an explicit formula for the local metrics of these spaces (Theorem 4.20). This allows to describe all non-symmetric curvature homogeneous spaces having the same curvature tensor as a fixed symmetric space. As an application of the results of this chapter, we show that every Einstein-like semi-symmetric space is locally symmetric (Theorem 4.22) and we give some immediate consequences of this result.

In Chapter 5, we introduce the notion of an *asymptotic foliation* in arbitrary dimensions (as a codimension one foliation) and we define planar, hyperbolic, parabolic and elliptic semi-symmetric spaces according to the number of asymptotic foliations they admit (infinitely many, two, one or none, respectively). Moreover, we show that the existence of an asymptotic foliation has strong implications on the form of the metric (Theorem 5.3), which simplifies considerably the basic system of partial differential equations. Further, we define the geometric notion of *algebraic rank* for a foliated semi-symmetric space as an integer $r \in \{0, 1, 2, 3, 4\}$. We then prove that the case $r = 0$ corresponds to a local product space $M^2 \times \mathbb{R}^n$ (Proposition 5.5) and we determine the possible values of the algebraic rank for a planar, hyperbolic, parabolic and elliptic semi-symmetric space (Proposition 5.6).

Chapter 6 is devoted to the study of three-dimensional, non-elliptic, foliated semi-symmetric spaces. The explicit formulas from [59] for “generic” cases are derived here once again, using the previous general

theory. But, besides that, all *non-generic* cases are also classified. The last material is based on the Doctoral Thesis by V. Hájková ([49]).

Chapter 7 contains a closer study of *asymptotically foliated semi-symmetric spaces*, i.e., non-elliptic foliated semi-symmetric spaces. For the case of algebraic rank equal to one, we are able to give a complete local classification by explicit metrics. The procedure is rather technical and consists mainly in applying the appropriate coordinate transformations. In the second section of this chapter, we then give families of examples having higher algebraic rank and covering all possibilities. (We mention that spaces of higher algebraic rank, as well as non-trivial planar spaces and hyperbolic spaces, are never complete.)

Chapter 8 deals with the remaining class of *elliptic* foliated semi-symmetric spaces. We present a general construction for these spaces when the algebraic rank is one. In particular, we show (as a local property) that an irreducible elliptic space of algebraic rank equal to one must be three-dimensional. Our treatment of these spaces is more uniform than that presented in [59]. We also give families of explicit examples of elliptic semi-symmetric spaces having higher algebraic rank and covering all possibilities.

In Chapter 9, we focus our attention on *complete* foliated semi-symmetric spaces. These have been studied in detail by Szabó in [139]. The first section of this chapter is therefore devoted to a short survey of his work, including a division of these spaces into three different types: trivial, parabolic and hyperbolic ones. (This terminology is different from that used in Chapters 5–8.) The spaces of trivial type are all local product spaces. As concerns the spaces of hyperbolic type, only one explicit example of this type was known until recently, namely the counterexample to Nomizu's conjecture by Takagi ([141]). This is realised as a hypersurface in $\mathbb{R}^4$. In order to find other examples, we first review Szabó's work on complete semi-symmetric hypersurfaces in Euclidean spaces ([140]). He classifies these hypersurfaces and describes general constructions for such spaces. Applying his procedure, we then extend Takagi's example to a whole family parametrised by holomorphic functions defined on the whole complex plane (Theorem 9.12). For the spaces of parabolic type, we prove an alternative geometric characterisation (Theorem 9.13) and, using this, we deduce

that the counterexamples to Nomizu's conjecture by Sekigawa ([119]) belong to this class. Finally, we clarify the relations between the spaces of trivial, parabolic and hyperbolic type in the sense of Szabó and the four classes of foliated semi-symmetric spaces (planar, hyperbolic, parabolic and elliptic) defined in Chapter 5.

In Chapter 10, we use the three-dimensional classification of Chapter 6 for the construction of hypersurfaces with type number two in $\mathbb{R}^4$. We obtain all possible situations anticipated by E. Cartan ([21]) as concerns the number of isometric deformations. In particular, by the embedding of certain "orthogonally hyperbolic spaces" one proves the existence of hypersurfaces admitting *exactly one* non-trivial isometric deformation. This result (belonging to V. Hájková ([49])) seems to be the first example of such a situation. (E. Cartan never proved the *actual existence* of these hypersurfaces. M. Dajczer, L. Florit and R. Tojeiro constructed such examples only very recently ([29]).) The last section presents the *intrinsic* classification of all ruled hypersurfaces in $\mathbb{R}^4$. (One obtains four different types of metrics, all written in explicit form.) Ruled hypersurfaces correspond to some "singular cases" of the classification from Chapter 6.

Chapter 11 is devoted to Riemannian manifolds of conullity two *modulo* some curvature tensor of constant sectional curvature c . The basic properties of foliated semi-symmetric spaces remain valid for these new classes of Riemannian manifolds. We restrict ourselves to the three-dimensional case and we give the classification of all generic asymptotically foliated spaces. The explicit formulas which we obtain are complete analogues of the corresponding formulas from Chapter 6. Examples of such spaces are hypersurfaces with type number two in S^4 and $\mathbb{H}^4$. This extended class of spaces (including the case $c = 0$) coincides, in dimension three, with the so-called "pseudo-symmetric spaces of constant type". A fair credit for this chapter is given to M. Sekizawa.

Chapter 12 is of an expository nature and is devoted to the recent results on curvature homogeneous spaces of non-symmetric type. (Those of symmetric type are treated in Chapter 4.) It includes a number of interesting isolated examples and also entire families of examples. The classification problem is solved, at least partially, in

dimension three. A basic theorem says that every algebraic curvature tensor in dimension three can be realised on a curvature homogeneous space (but not always on a locally homogeneous one). Moreover, if the prescribed curvature tensor is not of constant sectional curvature, then there is an infinite-dimensional space of examples of a given type which are not locally homogeneous. The topic treated in this chapter was initiated in 1960 by I. M. Singer ([131]) and there are still numerous open problems left for the readers, including the so-called Gromov conjecture. The chapter is dedicated to the memory of our friend and collaborator F. Tricerri. He contributed a lot to this field of Riemannian geometry and he originally planned to write a joint monograph with the last two authors about curvature homogeneous spaces. But he could never realise this idea because of his tragic death, together with his entire family, in an airplane crash at Mingdu in China on June 6, 1994.

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