CHARACTERIZATIONS OF INFORMATION MEASURES

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Dedicated by

Bruce Ebanks
to his wife Annalisa
and
his teacher Che Tat Ng,

and

Prasanna Sahoo
in memory of his father Hari
and
to his teacher Pallaniappan Kannappan,

and

Wolfgang Sander
to his wife Helda
and
daughters Michelle and Christin

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PREFACE

This book is a monograph on the characterizations of various measures of information. It has arisen from over fifteen years' research on the subject, much of which has been done by the authors. In this book we treat various properties of measures of information and the extent to which these properties characterize known measures of information. To the best of our knowledge this is the first book investigating this topic on open domains in depth; an earlier book by Aczél and Daróczy (1975) treated the subject on closed domains.

The book is written for students, educators, and practitioners of information theory, statistics, and functional equations. The goal of the book is twofold. The first goal is to provide the interested reader with up-to-date information about the characterization of information measures on open domains. The second is to present a comprehensive illustration of the use of functional equations to model and solve problems in information theory. Not many prerequisites are needed to understand this book. Some basic calculus, measure, probability and set theory facts, which can be found in almost all textbooks on these subjects, are sufficient.

We rely heavily upon the theory of functional equations, but we have presented in this book essentially everything that is needed in that regard. That is, the book includes all results about functional equations that we use in the characterizations of information measures. Indeed this book can be treated as a book on functional equations with applications to information theory. Additional information can be found in several books on functional equations, such as Aczél (1966), Kuczma (1985) or Aczél and Dhombres (1989). Another interesting feature is a detailed coverage of sum form information measures in a very general setting.

The book contains nine chapters. Chapter 1 includes some basic results from the theory of functional equations. In Chapter 2, we treat the branching property of information measures and discuss the characteristic functional equation for these measures. This chapter concludes with the symmetric solution of the cocycle equation. Chapter 3 presents some results regarding recursivity properties of information measures. Here, we treat recursivity in a very general form, the recursivity of multiplicative type. Chapter 4 deals with

the fundamental equation of information and presents the forms of recursive measures which satisfy some regularity property such as measurability.

The last five chapters cover the characterizations of sum form information measures which arise from various types of additivities. Chapter 5 presents, in addition to the classical additivity property of Shannon, two other more general types of additivity, known as additivity of type λ and additivity of type (M_1, M_2) . In this chapter the reader will also find a justification for the study of information measures with these additivity properties. Chapter 6 covers some basic sum form functional equations. These equations are instrumental in characterizations of all additive sum form information measures of type λ as well as all additive sum form information measures of type (M_1, M_2) . All additive sum form information measures with measurable generating function are explicitly determined in Chapter 7. Such measures are shown to be a linear combination of Kerridge's inaccuracy, entropies of degree one and zero, and a constant. In Chapters 8 and 9, the forms of all additive sum form information measures of type λ and of all weighted additive sum form information measures of type (M_1, M_2) are determined, respectively.

We do not claim that everything written in this book originated with us. In fact, we freely use the works of many researchers, and we have attempted to acknowledge all contributions. For some results we present new proofs which do not appear elsewhere. In other cases, we did not succeed in improving on the original explanations. Nonetheless we feel that it is worthwhile to have all these results gathered in one place and linked together in a coherent fashion. We hope the readers will agree.

Donald Knuth's $\text{\TeX}$ program was used to create the manuscript. We wish to thank Betty Hampton for her valuable assistance in typing a large part of the manuscript.

Bruce Ebanks, *Huntington*
 Prasanna Sahoo, *Louisville*
 Wolfgang Sander, *Braunschweig*

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