———— An Introduction to ————
Differential Geometry
and Topology in
Mathematical Physics

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An Introduction to Differential Geometry and Topology in Mathematical Physics

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**AN INTRODUCTION TO DIFFERENTIAL GEOMETRY AND
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Preface

The purpose of writing this book is to give an outline of the classical results and the most recent developments of differential geometry and topology, especially those important developments which are closely related to the new discoveries in the field of theoretical physics. The contents of this book consist of three parts:

In Part I, we introduce the fundamental knowledge of differential manifolds, including exterior differential, tangent space, cotangent space and their transformation properties. Differential operation of tensors on manifolds, Riemann manifold together with its conformal transformation, and complex manifolds are also introduced.

In Part II, the global topological properties of differential manifolds are discussed in details, including the theories of homotopy, homology, de Rham cohomology, fibre bundles and their characteristic classes which were first introduced by S. S. Chern.

In Part III, we emphasize the index theorem and the properties of simply connected 4-manifolds. Atiyah-Singer's index theorem and its applications, such as the derivation of the dimension of instantons' moduli space, are demonstrated. This part also provides an important theorem of Donaldson on intersection forms of 4-manifolds, followed by a very exciting conclusion drawn by M. H. Freedman and C. Taubes: "There is an uncountable family of open subsets of R^4_n which are all homeomorphic to R^4 , but no two are diffeomorphic." This is a very peculiar phenomenon not occurring in any R^n with $n \neq 4$, and it happens that the dimension of our universe is $n = 4$.

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