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Volume 3

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Vibration Analysis of Plates by the Superposition Method

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World Scientific

Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

VIBRATION ANALYSIS OF PLATES BY THE SUPERPOSITION METHOD

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ISBN 981-02-3681-6

This book is printed on acid-free paper.

Printed in Singapore by Uto-Print

*TO THE MEMORY
OF
MY PARENTS LAWRENCE AND ELLEN (ENRIGHT) GORMAN
AND THEIR BELOVED DAUGHTERS
BETTY AND AL*

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PREFACE

The subject of free vibration analysis of thin plates is one which extends well back into the last century. The development of metal alloys and other high strength materials permits the wide use of thin plates in the modern industrial world. In many instances, where structures are subjected to time-varying forces, the ability of designers to compute, in advance, the natural frequencies and mode shapes of these thin plates is of critical importance. This is true, not only in the aerospace industry, but applies to many civil structures as well. The range of industrial problems where plate free vibration analysis is required varies from design of offshore petroleum exploration equipment to thin electronic circuit boards.

A vast number of publications have appeared in the literature related to thin plate free vibration analysis. Nevertheless, historically, there have been two principal approaches taken toward resolving these problems. The most popular traditional approach is based on energy considerations and the method of analysis is known as the Rayleigh-Ritz method. In order to exploit this method one must choose an appropriate set of functions to represent the plate mode shape. These functions must also satisfy certain boundary conditions of the problem under study. Mode shapes are usually represented by an infinite series of functions, but only a finite number of functions can be employed in the analysis. It is found that frequencies thereby obtained generally represent upper limits for the actual frequencies.

An alternate approach to plate free vibration analysis has been the use of numerical methods to obtain solutions which satisfy, as closely as possible, the plate governing differential equation and the prescribed boundary conditions. There is no doubt that with the advent of the high-speed digital computer the use of finite element methods has become the most popular numerical approach. This method is widely utilized in modern industry.

During the past decade and a half the author and his associates have been involved in development and exploitation of a new analytical approach to rectangular plate free vibration analysis. The method of analysis is referred to as the "Superposition Method". One begins by superimposing a set of judiciously selected forced vibration problems for which exact series solutions are known. These are referred to as building blocks. They are usually driven by harmonic moments or vertical edge reactions distributed along one of the boundaries. With the building blocks superimposed, unknown Fourier driving coefficients appearing in their solutions are adjusted so as to satisfy the prescribed boundary conditions for the problem of interest.

The superposition technique provides a welcome alternative to the traditional Rayleigh-Ritz and finite element methods, and one which is becoming more and more attractive. Of the three techniques, only the superposition and Rayleigh-Ritz methods provide solutions based on continuum mechanics. Despite this shortcoming, the finite element method is usually resorted to when structures are so complicated that a solution based on continuum mechanics is virtually impossible to achieve.

Two of the most obvious advantages of the superposition method may be listed as follows:

1. It is the only one of the three methods which provides solutions which satisfy the governing differential equation exactly throughout the entire domain of the plate.
2. The problem of trying to select appropriate series of functions to represent plate lateral displacement when utilizing the Rayleigh-Ritz method often becomes a formidable one. In fact, in the case of plates with free edges, it has been demonstrated that fully appropriate functions do not exist. This is because mathematical formulation of free edge boundary conditions involves mixed spatial derivatives of the plate lateral displacement.

It is a great strength of the superposition method that no selection of such functions is required. In fact, the solution is self contained in that computed plate lateral displacements are generated from within and are based entirely on analytical solutions to the plate governing differential equation.

The objective set forth in this book is to make this highly accurate analytical method available to all potential users in the field of mechanical vibration. It is not the objective to include vast arrays of pre-computed eigenvalues. This has been taken care of by numerous publications in the existing literature which are referred to herein. It is almost certain that a much more general adoption of this powerful analytical technique, with its numerous advantages, has been delayed by the lack of a carefully prepared text explaining, in detail, how the method functions and how it can be exploited. This book is prepared with a view to filling this need.

The vast capabilities of the superposition method were not originally foreseen. It is found, for example, that in addition to rectangular plates it can be extended to handle straight edged plates, such as triangular plates, parallelogram plates, orthotropic plates, laminated plates, etc. All of these capabilities are described in the book.

In Chapter 1, the basic theory leading to solutions for rectangular plate free vibration problems is explained. Practical instruction is provided with regard to how best to prepare computer routines for efficient exploitation of this method. Solution to a typical free vibration problem is described in detail for illustrative purposes. Recognizing how easily the treatment of isotropic plate problems, as discussed in Chapter 1, can be extended to handle orthotropic plate problems, Chapter 2 is devoted to the analysis of orthotropic plates. Treatment of orthotropic plates is deliberately introduced at this early stage so that both isotropic and orthotropic plate theory can be applied to problems of the remaining chapters.

Experience has shown that the method is ideally suited for resolving problems of plates with elastic edge support. It also readily handles problems involving plates with point supports on their lateral surface and attached masses. These two families of problems are discussed in Chapters 3 and 4, respectively. Plates with mixed boundary conditions are considered in Chapter 5 while the problem of handling non-rectangular straight-edged plates is the subject of Chapter 6.

In Chapter 7 the resolution of problems with uniform in-plane forces is discussed in detail. It is shown how this family of problems is extended to include the rectangular plate buckling problem. Chapter 8 explains how the superposition method is readily exploited to handle Mindlin thick plate problems. The free vibration of composite plates is the subject of Chapters 9. Treatment of these topics by the superposition method is a relatively recent phenomenon, but it is found that, again, the method works quite well.

The author has found, however, that even though the traditional superposition technique works very well for the much more complicated vibration problems of thick Mindlin plates and laminated plates the amount of effort required on the part of the analyst is greatly increased. Accordingly, he has explored a slightly modified method for handling such problems. The only difference related to these modified approaches centres around the manner in which the basic "Building Block" solutions are obtained. Otherwise, the procedures are identical to those of the previous chapters. It is shown how this alternate approach employing the Galerkin procedure can lead to drastic reductions on the effort required by the analyst.

Chapter 10 is devoted to the determination of response of undamped rectangular plates to steady state harmonic excitation by lateral forces or bending moments. These forces or moments may be concentrated or distributed. It is explained how the material of the earlier chapters may be exploited to ensure that desired boundary conditions are satisfied.

Chapter 11, the final chapter, is left open for discussion of some interesting auxiliary problems amenable to solution by the superposition method. These are problems which do not quite fall into the categories of problems related to the previous chapters. It is explained, for example, how the superposition building block approach functions equally well when searching for stress functions related to general plane stress problems. Procedures for analysing rectangular plates resting on line supports of arbitrary configuration are described. It is also explained how the same procedures are employed to obtain rapid but quite accurate solutions to problems involving plates with discontinuities in edge support. The subject of analysing continuous multi-span plates is also introduced.

In each chapter results are tabulated for typical problems treated therein. In this way the reader will have accurate data against which his or her results may be compared. The ability to make such comparisons is essential so that the correct functioning of newly prepared computer routines can be verified.

Many people have contributed toward the realization of this book in one fashion or another. Their names are much too numerous to recognize here on an individual basis. Nevertheless, the author would like specifically to acknowledge the invaluable assistance of Dr. K.G. Gupta who prepared all of the original drawings for the figures appearing in the text. His critical attention to detail and innovative suggestions are deeply appreciated. The author would like also to recognize the vital contribution of Ms. Josée Rocheleau who typed all of the manuscript.

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NOMENCLATURE

Listing of frequently used symbols

a, b	Rectangular plate edge lengths
D	Plate flexural rigidity = $Eh^3/12(1 - \nu^2)$
E	Young's modulus of elasticity
f	Plate vibration frequency, H_z
G	Modulus of elasticity in shear
h	Plate thickness
K	Number of terms in series expansions
k	Basic lateral elastic stiffness
K_L	Dimensionless lateral elastic stiffness
K_R	Dimensionless rotational elastic stiffness
K^*	Dimensionless local spring stiffness
M	Bending moment
M_{xy}	Twisting moment
M_x, M_y	Moments associated with x and y , directions
M_ξ, M_η	$M_x a/D$, and $M_y b^2/aD$, respectively.
M_o	Concentrated harmonic bending moment
M_R	Ratio of attached mass to mass of plate
N_x, N_y	In-plane normal stress resultants
N_{xy}	In-plane shear stress resultant
P	Amplitude of harmonic concentrated force
P^*	$-2P b^3 / a^2 D$
P_x	Plate total in-plane force in " x " direction
P_ξ	$P_x b/D$
Q_x, Q_y	Shear forces associated with x , and y , directions
S	Plate edge length-to-thickness ratio, a/h
T	Basic torsional elastic stiffness
u, v	Coordinates of plate point support
u^*, v^*	$(1 - u)$, and $(1 - v)$
V	Plate vertical edge reaction
V_ξ	$V a^2 / D$
V_η	$V b^3 / a D$
V_o	Concentrated harmonic edge reaction

W	Plate lateral displacement divided by "a"
x,y	Basic plate coordinates
z	Coordinate running downward from plate mid-plane

Symbols associated with Orthotropic Plates

D_x, D_y	$E_x h^3 / [12 (1 - \nu_x \nu_y)], E_y h^3 / [12 (1 - \nu_x \nu_y)]$
DHX, DHY	H/D_x , and H/D_y
D_t	$G_{xy} h^3 / 12$
E_x, E_y	Young's Moduli associated with x, and y directions
G_{xy}	Elastic modulus in shear associated with x, y plane
2H	$\nu_y D_x + \nu_x D_y + 4 D_t$

Greek Symbols

ξ, η	Dimensionless coordinates, x/a and y/b
ν	Poisson's ratio
ν_x, ν_y	Poisson ratios associated with x, and y directions
ν_x^*	$= 2 H/D_y - \nu_y$
ν_y^*	$= 2 H/D_x - \nu_x$
ν^*	$= (2 - \nu)$
ν_1	$= (1 - \nu) / 2$
ν_2	$= (1 + \nu) / 2$
ν_3	$= 6 \kappa^2 (1 - \nu)$
Ψ_ξ, Ψ_η	Plate cross-section rotations associated with ξ , and η , directions
ϕ	Plate aspect ratio, b/a
ϕ_h	Plate thickness ratio, h/a
κ^2	Mindlin shear correction factor
ω	Circular frequency of free vibration
ω_d	Circular frequency of forced vibration excitation
ϵ_x, ϵ_y	Linear strain related to x, and y, directions
λ^2	Eigenvalue $= \omega^2 a^2 \sqrt{\rho/D}$
ρ	Mass of plate per unit area
σ_x, σ_y	Normal stresses in x, and y, direction

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