LECTURES ON REPRESENTATION THEORY

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JING-SONG HUANG

*Department of Mathematics
The Hong Kong University of Science and Technology
Hong Kong*

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Preface

In this book we expand upon a series of lectures delivered during the Nankai Mathematical Summer School. The purpose of those lectures was to provide an introduction to Lie groups, Lie algebras and their representations as well as to introduce some directions of current research to graduate students who have little specialized knowledge of representation theory.

As we move forward into the next century, science has become essential in every part of our society. Mathematics is the very heart of science because all scientific truth is expressed in mathematical terms. Group symmetry is considered to be the soul of modern mathematics. Representation theory is a fundamental tool for studying group symmetry – geometric, analytic, or algebraic – by means of linear algebra. It plays an important role in many recent developments of mathematics and theoretical physics. Indeed, in his often-quoted statement, Izrail Gelfand remarked, “all of mathematics is some kind of representation theory.”

Albert Einstein once said, “everything should be made as simple as possible, but not simpler.” We think this should be particularly true for mathematical theories. Therefore we assume here that the reader only has a good knowledge of undergraduate linear algebra and a modest knowledge of undergraduate abstract algebra. The book is organized in such a way that we think it will be easier for a beginner to understand. The book is divided into four parts: (I) finite groups, (II) simple Lie algebras, (III) compact Lie groups and (IV) non-compact Lie groups. Each part consists of three chapters among which the first one is about the structure theory of the corresponding subject, the second one introduces the general theory of the corresponding representations and the third one considers some special, but in my opinion, the most interesting topics of representation theory. The similarities and connections between all these four different parts are quite obvious.

Most of Part I-III of the book consists of relatively elementary material of representation theory. These theories are natural continuation of basic linear algebra and the reader will find applications in many fields of mathematics and physics. We have not attempted to give the most complete proofs but to convey the essential ideas of representation theory. Useful references are provided so

that the interested readers can find the detailed proofs elsewhere. Part I deals with finite groups which do not require any Lie theory. We use the symmetric group S_n as an example to show the characters of group representations. Part II is about the structure and representation theory of simple Lie algebras over $\mathbb{C}$. Part III considers the representations of compact Lie groups and applications. The last chapter of Part III contains recent results on extension of Weyl's constructions. Part IV covers some advanced material on infinite-dimensional representations of non-compact groups. We use the standard example, the representations of $SL(2, \mathbb{R})$. We have also included the theory of coadjoint nilpotent orbits and minimal representations of complex simple groups. An alternative approach to an introduction of infinite-dimensional representations that was presented in the lectures at Nankai University is the representations of the general Linear group $GL(n, \mathbb{R})$ and its universal covering group $\widetilde{GL}(n, \mathbb{R})$. The theory of complete classifications of irreducible unitary representations of $GL(n, \mathbb{R})$ and of $\widetilde{GL}(n, \mathbb{R})$ together with the lifting of the unitary characters which relates the two groups is extremely appealing. However, we have included these materials here to keep this introductory book a reasonable length. Interested readers can refer to three journal articles [V2], [H1] and [AH]. The last part of the book, though short, touches on almost all the important ideas of current research such as orbit methods, minimal representations and dual pair correspondences.

In his famous lecture delivered at the International Congress of Mathematicians in 1900, David Hilbert quoted an unnamed mathematician, "a mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." From this point of view, the representation theory is not yet complete, but it is an exciting field that has been attracting many young and enthusiastic people who will be exploring it continuously for many years, well into the 21st century.

I would like to thank the mathematicians I met at Nankai University and Nankai Institute of Mathematics, especially Hou Zixin, Zhou Xinwei and Liang Ke, for their hospitality during the summer school. Thanks are also due to Hu Sen, Lin Yang Qing, Tang Wen Jing and Wang Shu Juan of World Scientific Publishing for their professional help. Most of all, I am very grateful to David Vogan and Roger Zierau for reading the first draft of the manuscript and offering many great ideas for improvements.

Jing-Song Huang

HKUST
Clear Water Bay
Hong Kong

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