GENERALIZED —

ASSOCIATED —

— LEGENDRE FUNCTIONS —

AND THEIR APPLICATIONS —

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GENERALIZED ASSOCIATED LEGENDRE FUNCTIONS AND THEIR APPLICATIONS

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Foreword

In many areas of applied mathematics and in particular, in applied analysis and differential equations various types of special functions become essential tools for scientists and engineers. They appear in the solutions of initial-value or boundary-value problems of mathematical physics and usually satisfy certain classes of ordinary differential equations. One of the important classes of special functions is of hypergeometric type. It includes all classical hypergeometric functions as the well-known Gaussian hypergeometric function, Bessel, Macdonald, Neumann, Legendre, Whittaker, Lommel, Thompson, Kummer, Tricomi, Appel, Wright etc. and the generalized hypergeometric functions ${}_pF_q$, E -function of Mac-Robert, Meijer's G -function and Fox's H -function.

As is known the hypergeometric functions can be defined through the contour integrals of the Mellin-Barnes type and some of them properties and asymptotic representations can be derived using the theory of the Mellin transform. This approach enables also to study integral transformations whose kernels depend upon variables and parameters of the special functions which are involved in them.

This book is devoted to the theory and applications of the **generalized associated Legendre functions** of the first and the second kind $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, which are important representatives of the hypergeometric functions. They occur as generalizations of classical Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ of the first and the second kind respectively. The authors use various methods of contour integration, consider solutions of the corresponding differential equations in order to obtain important properties of the generalized associated Legendre functions as them series representations, asymptotic formulas in a neighborhood of singular points, zeros properties, connection with the Jacobi functions, relations with Bessel functions, elliptic integrals and incomplete beta functions.

Some of the interesting aspects, which are considered here involve classes of dual and triple integral equations being associated with the function $P_{-1/2+i\tau}^{m,n}(\cosh \alpha)$. This function represents the kernel of the generalized Mehler-Fock transform of order (m,n) ,

$$M_{m,n}(\tau) = \int_0^\infty f(x) P_{-1/2+i\tau}^{m,n}(\cosh x) \sinh x dx,$$

which, in turn, is one of the basic integral transformations (the so-called *index transforms*) depending upon a parameter of the Legendre function. It should be remarked that the authors consider fractional integro-differential properties of the generalized associated Legendre functions and give various generalizations of Buschman-Erdelyi's type integral operators.

This book also presents the theory of factorization and composition structure of integral operators associated with generalized Legendre functions, which has important applications in finding solutions of the corresponding integral equations. The method of the considered integral equations gives series of examples of integrals over variables and parameters of the generalized Legendre functions. One of such sources of formulas is served by the pair of the following Mehler-Fock transforms

$$g(x) = \pi^{-1} x^{-1/2} (1+x)^{\mu/2} \int_0^\infty \tau \sinh(\pi\tau/2) P_{(i\tau-1)/2}^\mu \left(\frac{2}{x} + 1 \right) f(\tau) d\tau, \quad x > 0,$$

$$f(\tau) = \frac{1}{2} |\Gamma(1/2 - \mu - i\tau/2)|^2 \int_0^\infty (1+x)^{-\mu/2} P_{(i\tau-1)/2}^\mu \left(\frac{2}{x} + 1 \right) g(x) \frac{dx}{x^{3/2}},$$

and the Parseval equality

$$\int_0^\infty (1+x)^{-\mu} |g(x)|^2 \frac{dx}{x} = \frac{2}{\pi} \int_0^\infty \frac{\tau \sinh(\pi\tau/2)}{|\Gamma(1/2 - \mu - i\tau/2)|^2} |f(\tau)|^2 d\tau.$$

Furthermore, recently it was discovered new pair of reciprocal index transforms over the upper index of the generalized Legendre function of the first kind, namely,

$$\begin{aligned} g(x) &= \frac{(1-x)^{-1/2}}{4\sqrt{\pi}} \int_0^\infty \tau \Gamma(1-i\tau) \sinh(\pi\tau) \\ &\times [P_0^{i\tau}(\sqrt{1-x}) + P_0^{i\tau}(-\sqrt{1-x})] f(\tau) d\tau, \end{aligned}$$

where $0 < x < 1$,

$$f(\tau) = \frac{2i\Gamma(i\tau)}{\sqrt{\pi} \sinh(\pi\tau)} \int_0^1 [P_{-1}^{-i\tau}(\sqrt{1-x}) + P_{-1}^{-i\tau}(-\sqrt{1-x})] g(x) \frac{dx}{x},$$

which can be used to evaluate a variety of improper integrals associated with the combination of Legendre functions.

One of the applications of general special functions, which is also worth mentioning here is to consider integral operators of the *Mellin convolution type* with the generalized hypergeometric functions as them kernels from the semigroup theoretical point of view. Indeed, let us introduce for all $0 < \alpha < 1$ the following Wright function

$$\Phi_\alpha(y) = \sum_{n=0}^\infty \frac{(-y)^n}{n! \Gamma(1 - \alpha - \alpha n)}.$$

First, it is known that $\Phi_\alpha(y) \geq 0$, when $y \in \mathbf{R}_+$. Second, from the theory of the Mellin transform it follows the representation for $\Phi_\alpha(y)$ in terms of the Mellin-Barnes integral of the ratio of Euler's Gamma functions

$$\Phi_\alpha(y) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(s)}{\Gamma(1-\alpha+\alpha s)} y^{-s} ds.$$

For $0 < \alpha < 1$ the latter integral is absolutely convergent via the asymptotic behavior at the imaginary infinity of the Gamma function. Similar its convergence can be motivated in L_2 -sense in view of the Parseval equality for the Mellin transform

$$\int_0^\infty |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^\infty |f^*(1/2 + it)|^2 dt,$$

where

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx$$

is the Mellin transform of the function f in the space $L_2(0, \infty)$.

Hence one can consider the following integral operator of the Mellin convolution type, which contains as the kernel the Wright function Φ_α

$$(G_\alpha f)(x) = \frac{1}{x^\alpha} \int_0^\infty \Phi_\alpha\left(\frac{y}{x^\alpha}\right) f(y) dy.$$

Applying the Parseval equality one can deduce the representation of the latter convolution operator in terms of the Mellin transforms as

$$(G_\alpha f)(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(s)}{\Gamma(1-\alpha+\alpha s)} f^*(1-s) x^{\alpha(s-1)} ds.$$

From the inversion theorem for the Mellin transform we immediately obtain as a corollary that $(G_1 f) = f$. Moreover, we can extend now the range of α to $\alpha > 0$ by assuming certain asymptotic behavior at infinity for the Mellin transform. Then the latter integral is convergent in appropriate sense and operator G_α is defined for all $\alpha > 0$. Hence considering the composition of two operators G_α, G_β we can prove the following semigroup property

$$(G_\alpha G_\beta f)(x) = (G_{\alpha\beta} f)(x)$$

and deduce the form of the inverse operator as $(G_\alpha^{-1} f)(x) = (G_{1/\alpha} f)(x)$. This variety of operators G_α forms uniformly continuous semigroup in appropriate Banach space in view of its definition by substitution $\alpha = e^\xi, \xi \geq 0$. Furthermore we can find the infinitesimal generator for this semigroup. It gives the following operator

$$\begin{aligned} (Yf)(x) &= \lim_{\alpha \rightarrow 1} \frac{(G_\alpha f)(x) - f(x)}{\alpha - 1} \\ &= x(\log x + \gamma - 1) \frac{df}{dx} + \gamma f(x) - \frac{d}{dx} \int_0^x \frac{f(x) - f(y)}{x - y} y dy, \end{aligned}$$

where γ is the Euler constant.

Thus we have demonstrated an example of the Mellin type convolution operator with the Wright function as the kernel showing the importance of generalized hypergeometric functions in various applications.

Finally as a conclusion, I am happy to recommend this presentation of generalized associated Legendre functions by N. Virchenko and I. Fedotova to all graduate students and researchers, other users of special functions, integral transformations, differential equations in the mathematical, physical and engineering sciences who may find and develop interesting methods and ideas for their works.

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Preface

Each mathematician or engineer who practically works with differential equations is familiar with special functions. A great number of papers have been devoted to special functions. Nevertheless, their further study is very interesting and useful for different branches of science.

Essentially, special functions appear under solving some boundary value problems of mathematical physics (that are solved by using method of separation of variables), in investigation of series, etc. The diversity of the problems generating special functions has led to a quick increase in a number of functions used in applications from the simplest transcendental functions to the hypergeometric functions, the Mathieu functions, etc. It should be noted that evaluation of the importance of one or another class of special functions has changed substantially during the last hundred years. The elliptic integrals and related functions turned out to be the most interesting in the past century, but further development of mathematics has putted in the forefront another class — the functions of the hypergeometric type or the so-called functions of mathematical physics. The hypergeometric function and its special and degenerated cases (the Bessel and the Legendre functions, classical orthogonal polynomials such as Jacobi, Laguerre, Hermite polynomials etc.) play an increasing part in various sections of pure and applied mathematics.

Special functions are kernels of many integral transforms. In turn, the theory of integral transforms is an effective mathematical instrument for solving many problems in analysis and the theory of partitions, for researching boundary value problems of the mathematical physics, etc. The Fourier, Laplace, Mehler-Fock, Mellin transforms can be applied to many problems important for physics. Some new integral transforms have been introduced recently.

Application of new special functions allows us to increase considerably the number of problems which can be solved in a closed form, to examine these solutions, to investigate relationships between different classes of special functions.

Special functions have been studied by many scientists, and N.N. Lebedev [72], E.T. Whittaker and G.N. Watson [141], A. Kratzer and W. Franz [55], H. Bateman and A. Erdélyi [23-26], E.W. Hobson [46], F.W.J. Olver [104], N. Ja. Vilenkin [125], I.S. Gradshteyn and J.M. Ryzhik [43], A.P. Prudnikov and Yu.A. Brichkov [109-111], E. Yanke, F. Emde and F. Lösch [146], M. Abramovitz and I.A. Stegun [1] are among them.

One of the most important special functions is the Gauss hypergeometric function ${}_2F_1(a, b; c; z) \equiv F(a, b; c; z)$ that can be determined by the expression

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (1)$$

where $(a)_n$ is the Pochhammer symbol:

$$(a)_0 = 1, (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

If $c \neq 0, -1, -2, \dots$, then ${}_2F_1$ satisfies the following differential equation:

$$z(z-1) \frac{d^2 u}{dz^2} + [c - (a+b+1)z] \frac{du}{dz} - abu = 0, \quad (2)$$

This solution is regular in the point $z=0$. The series converges at $z=-1$ and $z=1$ provided that $c-a-b+1 > 0$ and $c-a-b > 0$ respectively.

Various investigations of the Gauss function have been given by many authors. Studying properties of this class of functions is essential for solving the so-called Fuchian equations with three critical points. It should be noted that many algebraic or special functions that occur in problems of applied mathematics can be expressed in terms of the hypergeometric function. The Bessel, Legendre, Whittaker and other special functions, and the classical orthogonal polynomials are particular cases of the hypergeometric function or their various combinations. The Gauss function would apply in astronomy, mechanics, mathematical physics, etc.

The Legendre functions are the most well known particular cases of the hypergeometric function. They have been discovered by Laplace and Legendre as early as in the 18th century. Later on their importance has grown substantially due to their connections with many problems of mathematical physics.

For example, Legendre functions occur in solving the wave equation in the spherical coordinates r, φ, θ . Laplace operator ΔV for spherical coordinates is of the form

$$\Delta V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 V}{\partial \varphi^2},$$

and a solution of the equation $\Delta V = 0$ can be represented as

$$V = R(r) T(\theta) F(\varphi),$$

where $T(\theta)$ satisfies the ordinary differential equation

$$\frac{d^2 T}{d\theta^2} + \operatorname{ctg} \theta \frac{dT}{d\theta} + \left[\nu(\nu+1) - \frac{\mu^2}{\sin^2 \theta} \right] T = 0.$$

Here μ and ν are arbitrary constants arising under separation of variables. Substituting $x = \cos \theta$ into this equation we receive the Legendre differential equation

$$(1-x^2) \frac{d^2 T}{dx^2} - 2x \frac{dT}{dx} + \left[\nu(\nu+1) - \frac{\mu^2}{1-x^2} \right] T = 0 \quad (3)$$

which is a particular case of (2). Two linearly independent solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ of (3) are called the Legendre functions of the first and the second kind, respectively. They can be expressed in different ways in terms of the hypergeometric function.

The Legendre equation arises in the potential theory for spheroidal, toroidal and other coordinates.

The properties of $P_n^m(z)$ and $Q_n^m(z)$ with integral values of n and m have been studied in the works of Heine, Hobson and others.

In particular, Hobson [46] gave a definition for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ for unrestricted values of ν, μ, z by means of a contour integral by using Pochhammer and Jordan double contour integrals.

The method of contour integrals is suitable for investigating $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ because it does not require convergence of infinite integrals.

A general theory of the Legendre functions has been developed also by Burns [55], who has used the contour integrals with Γ -functions for representation of the hypergeometric functions. A number of boundary value problems whose solutions contain the Legendre functions have been examined, for example, in [8, 44, 47, 72, 106, 108].

The Legendre functions are used as kernels of many integral transforms. There are the Mehler-Fock transform [39, 88] and several its generalizations [77, 103, 144]. These integral transforms allow us to obtain explicit solutions of many plane and space problems of the theory of elasticity, of mathematical physics, etc.

The Mehler-Fock type transforms are suited in the theory of dual, triple or summary equations involving the Legendre functions.

There exist various generalizations of the Legendre and the associated Legendre functions. In 1957, Kuipers and Meulenbeld admitted into consideration the generalized associated Legendre functions of the first and the second kinds $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, respectively. These functions satisfy the differential equation

$$\frac{d}{dz} \left[(1-z^2) \frac{du}{dz} \right] + \left[k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)} \right] u = 0 \quad (4)$$

that for $m = n$ yields the Legendre differential equation. Before, this equation has been considered only for real values of the variable z such that $-1 < z < 1$, among others by H. Bateman [6] in connection with a treatment of the homogeneous wave equations. The equation (4) appears also in applying the method of Zernike and Brinkman [148] to the solution of the general Laplace equation of higher dimensions. D.J. Hofsommer [62] constructed two solutions of (4), where z is real such that $-1 < z < 1$, and he derived many properties of these solutions under certain conditions for k, m and n . But the main results in the theory of the generalized associated Legendre functions have been obtained by Kuipers and Meulenbeld, and that is why these functions are called Kuipers-Meulenbeld functions.

In [62], Kuipers and Meulenbeld have defined two linearly independent solutions of (4) valid for unrestricted values of k, m, n . The results have been expressed in terms of contour integrals and the hypergeometric functions. These solutions give rise to the definition of a class of functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ which for $m = n$ are well known associated Legendre functions $P_k^m(z)$ and $Q_k^m(z)$. Some particular expansions for various values of k, m and n have been obtained too.

Later on a number of papers of these authors has been devoted to the generalized associated Legendre functions. Namely, some recurrence formulae for $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ and their derivatives were given in [58, 59, 64, 90, 91, 92]. The analytical expansions have been obtained in [63, 65], and the expression for wronskians has been found in [92]. The definitions of $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ for $z = \cos \theta \in (-1, 1)$ have been given in [89]. The results of [58, 60, 65, 68] include some integral representations for the generalized associated Legendre functions. A few generalizations of the well-known formulas of Heine, Neumann and others have been found in [61, 93, 118]. A list of some integrals involving the generalized associated Legendre functions have been presented in [60, 96].

Expressing the Jacobi functions of the first and the second kinds $P_k^{(\alpha,\beta)}(z)$ and $Q_k^{(\alpha,\beta)}(z)$ in terms of the hypergeometric functions, we see that there exists a close relation between these functions and the generalized associated Legendre functions provided the parameters k, m, n and z satisfy certain conditions. Many properties of the Jacobi functions continue to hold for the generalized associated Legendre functions under a certain restriction on k, m, n . The formulas connecting these two classes of special functions have been given, for example, in [60, 95]. Some theorems on integral representations of the generalized associated Legendre functions (by using the known integrals involving the Jacobi polynomials), the orthogonality relations for $P_k^{m,n}(z)$ and $P_l^{m,n}(z)$, $k \neq l$, the generalized Rodrigues formula, the generating functions for particular values of k, m and n were obtained in [60, 95].

The relationship between the generalized associated Legendre functions, elliptic integrals of the first and the second kinds and the incomplete B-function

$$B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt \quad (5)$$

have been established in [94] for special values of parameters. Expansions of $Q_k^{m,n}(z)$ in terms of the Mac-Robert functions were established in [66].

The integral transform involving $P_k^{m,n}(z)$ and asymptotic behavior of $P_k^{m,n}(\operatorname{ch} \alpha)$ and $P_k^{m,n}(\cos \alpha)$ for large k were studied by Götze [42].

Braaksma and Meulenbeld proved in [9] a number of integral theorems. By using the generalized associated Legendre function of the first and the second kinds as kernels they defined the generalized Mehler-Fock integral transforms and some their special cases. It is of interest to note the book [83], where authors have collected the various generalized Mehler-Fock transforms, occurring in the literature.

Some results connected with the generalized associated Legendre functions can be found in the papers of authors of this book. Our interests in this problem lie in such domains as an asymptotic behavior of functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, derivatives of these functions [28, 37, 126, 131], the integral representations [28, 31, 33, 36, 126, 131, 133, 136], connection of the generalized Legendre functions with other special functions [35, 36], their applications to the theory of integral transforms [29, 127, 129, 137, 138], and to the theory of dual integral equations [127-129, 132, 138-140].

A study of the generalized associated Legendre functions allows us significantly extend a number of problems and represent their solutions in a closed form. In particular, some partial differential equations whose solutions contain $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ are considered in [131]. For example, in the case of the prolate ellipsoid of rotation the system of the prolate elliptic coordinates have the following form:

$$x = c \operatorname{sh} \alpha \sin \beta \cos \varphi,$$

$$y = c \operatorname{sh} \alpha \sin \beta \sin \varphi,$$

$$z = c \operatorname{ch} \alpha \cos \beta,$$

where c is the scale multiplier, $0 \leq \alpha \leq \infty$, $0 \leq \beta \leq \pi$, $-\pi \leq \varphi \leq \pi$, and the Lamé coefficients are

$$H_\alpha = H_\beta = c \sqrt{\operatorname{sh}^2 \alpha + \sin^2 \beta}, H_\varphi = c \operatorname{sh} \alpha \sin \beta.$$

Let us consider in these coordinates the following equation:

$$\frac{1}{\operatorname{sh} \alpha} \frac{\partial}{\partial \alpha} \left(\operatorname{sh} \alpha \frac{\partial u}{\partial \alpha} \right) + \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial u}{\partial \beta} \right) + \left(\frac{1}{\operatorname{sh}^2 \alpha} + \frac{1}{\sin^2 \beta} \right) \frac{\partial^2 u}{\partial \varphi^2} - \left(\frac{\cos \beta}{\sin^2 \beta} + \frac{\operatorname{ch} \alpha}{\operatorname{sh}^2 \alpha} \right) \frac{\partial u}{\partial t} = 0. \quad (6)$$

We find the solution in the form

$$u = A(\alpha)B(\beta)\Phi(\varphi)T(t). \quad (7)$$

Further, we separate the variables

$$\begin{aligned} & \left\{ \left[\frac{1}{A \operatorname{sh} \alpha} \frac{d}{d\alpha} \left(\operatorname{sh} \alpha \frac{dA}{d\alpha} \right) - \frac{\operatorname{ch} \alpha}{\operatorname{sh}^2 \alpha} \frac{1}{T} \frac{dT}{dt} \right] + \right. \\ & \left. + \left[\frac{1}{B \sin \beta} \frac{d}{d\beta} \left(\sin \beta \frac{dB}{d\beta} \right) - \frac{\cos \beta}{\sin^2 \beta} \frac{1}{T} \frac{dT}{dt} \right] \right\} \left(\frac{1}{\operatorname{sh}^2 \alpha} + \frac{1}{\sin^2 \beta} \right) = \\ & = - \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2}. \end{aligned} \quad (8)$$

The equality is possible only for the case when the both sides are equal to the same constant. We denote this constant by $\mu^2 = \frac{m^2 + n^2}{2}$ and introduce the constant $\frac{m^2 - n^2}{2}$ for separating t . It is easy to show that the factors of expression (7) satisfy the following ordinary differential equations:

$$\frac{d^2 \Phi}{d\varphi^2} + \frac{m^2 + n^2}{2} \Phi = 0,$$

$$\frac{1}{\operatorname{sh} \alpha} \frac{d}{d\alpha} \left(\operatorname{sh} \alpha \frac{dA}{d\alpha} \right) - \left[k(k+1) - \frac{m^2}{2(1 - \operatorname{ch} \alpha)} - \frac{n^2}{2(1 + \operatorname{ch} \alpha)} \right] A = 0, \quad (9)$$

$$\frac{1}{\sin \beta} \frac{d}{d\beta} \left(\sin \beta \frac{dB}{d\beta} \right) + \left[k(k+1) - \frac{m^2}{2(1 - \cos \beta)} - \frac{n^2}{2(1 + \cos \beta)} \right] B = 0, \quad (10)$$

$$\frac{dT}{dt} + \frac{n^2 - m^2}{2} T = 0.$$

The parameters k , m and n must be chosen by the concrete boundary conditions. The solutions of (9) and (10) can be expressed in terms of $P_k^{m,n}(\text{ch } \alpha)$, $Q_k^{m,n}(\text{ch } \alpha)$, $P_k^{m,n}(\cos \alpha)$ and $Q_k^{m,n}(\cos \alpha)$. For the case of the toroidal coordinates α, β, φ we have

$$\begin{aligned} x &= \frac{c \text{ch } \alpha \cos \varphi}{\text{ch } \alpha - \cos \beta}, \\ y &= \frac{c \text{sh } \alpha \sin \varphi}{\text{ch } \alpha - \cos \beta}, \\ z &= \frac{c \sin \beta}{\text{ch } \alpha - \cos \beta}, \end{aligned}$$

where c is a scale factor, $0 \leq \alpha < \infty$, $-\pi \leq \beta \leq \pi$, $-\pi < \varphi \leq \pi$ let us consider the differential equation

$$\begin{aligned} &\frac{\partial}{\partial \alpha} \left(\frac{\text{sh } \alpha}{\text{ch } \alpha - \cos \beta} \frac{\partial u}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{\text{sh } \alpha}{\text{ch } \alpha - \cos \beta} \frac{\partial u}{\partial \beta} \right) + \\ &+ \frac{1}{(\text{ch } \alpha - \cos \beta) \text{sh } \alpha} \frac{\partial^2 u}{\partial \varphi^2} - \frac{\text{ch } \alpha}{(\text{ch } \alpha - \cos \beta) \text{sh } \alpha} \frac{\partial u}{\partial t} = 0. \end{aligned} \quad (11)$$

After putting $u = \sqrt{2(\text{ch } \alpha - \cos \beta)} v$, (11) becomes as

$$\frac{\partial^2 v}{\partial \alpha^2} + \frac{\partial^2 v}{\partial \beta^2} + \text{cth } \beta \frac{\partial v}{\partial \alpha} + \frac{1}{4} v + \frac{1}{\text{sh}^2 \alpha} \frac{\partial^2 v}{\partial \varphi^2} - \frac{\text{ch } \alpha}{\text{sh}^2 \alpha} \frac{\partial v}{\partial t} = 0. \quad (12)$$

Put $v = A(\alpha)B(\beta)\Phi(\varphi)T(t)$. By the procedure of separation of variables we obtain the following equation for $A(\alpha)$:

$$\frac{1}{\text{sh } \alpha} \frac{d}{d\alpha} \left(\text{sh } \alpha \frac{dA}{d\alpha} \right) - \left[k(k+1) - \frac{1}{4} - \frac{m^2}{2(1 - \text{ch } \alpha)} - \frac{n^2}{2(1 + \text{ch } \alpha)} \right] A = 0.$$

Its solutions are $P_k^{m,n}(\text{ch } \alpha)$ and $Q_k^{m,n}(\text{ch } \alpha)$.

Some other partial differential equations leading to the generalized associated Legendre functions are

$$\Delta u + \frac{z}{(x^2 + y^2)\sqrt{x^2 + y^2 + z^2}} \frac{\partial u}{\partial t} = 0, \quad (13)$$

$$\Delta u - \frac{\operatorname{ch} \alpha}{\operatorname{sh}^2 \alpha} \frac{\partial^2 u}{\partial t^2} = 0, \quad (14)$$

$$\Delta u - \frac{\operatorname{ch} \alpha}{\operatorname{sh}^2 \alpha} \frac{\partial u}{\partial t} + \rho^2 u = 0. \quad (15)$$

It should be noted that one of the methods of generalization of famous special functions consists in using generalized and modified hypergeometric functions. The generalized hypergeometric functions ${}_pF_q$ in [24] is given by expression

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad p \leq q+1. \quad (16)$$

E-functions of Mac-Robert, G-functions of Meijer, H-functions of Fox are of the same type. They are used in the theory of integral equations and as kernels of integral transforms [24].

Wright [143] has discussed the asymptotic behavior of the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 n) \cdots \Gamma(a_p + \alpha_p n)}{\Gamma(b_1 + \beta_1 n) \cdots \Gamma(b_q + \beta_q n)} \frac{z^n}{n!}$$

for large $|z|$. This is a class of functions still more general than ${}_pF_q$. In the case $a_p = 1, b_q = 1$ this series becomes ${}_pF_q$.

Simple example of Wright's hypergeometric function is Wright's generalized Bessel function. The asymptotic expansion of this particular function was studied in details in [142].

Wright's generalized Legendre functions have been defined and their integral representations have been given in [21]. Wright's generalization of the functions of Kuipers and Meulenbeld was discussed in [35].

The present book is devoted to the generalized associated Legendre functions. For the first time we try to collect and unify the information about these functions from various papers and notations. The new results have been obtained and formulated in the form of theorems, lemmas and examples. The corresponding calculations are often omitted. Since we require some results from the theory of hypergeometric functions, adopted to our purpose, we collect them in appendix.

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