QUANTUM INVARIANTS

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QUANTUM INVARIANTS

A Study of Knots, 3-Manifolds, and Their Sets

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Preface

In the 1980s low-dimensional topology encountered mathematical physics, and their interactions yielded infinitely many new invariants, called quantum invariants, of knots and 3-manifolds. This monograph provides an extensive and self-contained presentation of quantum and related invariants. This wealth of invariants gives us a new viewpoint in 3-dimensional topology: a study of the sets of knots and 3-manifolds, as well as a study of individual knots and 3-manifolds.

This monograph is addressed to readers with a basic knowledge of topology, such as homology groups and fundamental groups, as well as some acquaintance with basic abstract algebra.

When we tie a strand, a knot appears. Since ancient times many knots have been invented and contrived for practical and decorative purposes (see [TuGr96]). Tying strands in different ways, different knot types appear, albeit with significant redundancy. Thus we are led to ask: how various are the knot types? From the mathematical viewpoint knots are quite complicated and fascinating objects, even when we study them with all the mathematical approaches developed so far. The study of knots and knot types as mathematical objects is called *knot theory*.

There were attempts, say [Tai1898] (see also [TuGr96, Prz92, Epp99] for expositions of the history of knot theory), to classify knot types in the 19th century, though the mathematical fundamentals required to proceed with the classification rigorously were far from sufficiently developed in that age. This situation was changed by the creation of the discipline of topology at the beginning of the 20th century. Mathematical fundamentals of topology, such as homology groups, were established in the early 20th century, and subsequently "classical" knot theory, based on algebraic topology, developed rapidly (see [CrFo77, Rol90] for expositions of "classical" knot theory). The Alexander polynomial, discovered in the 1920s, was one of main achievements in "classical" knot theory. This polynomial is an isotopy

invariant of knots, where by an *isotopy invariant* we mean a map from the set of knot types to a well-known set, such as a polynomial ring.

A dramatic transformation of the study of knot invariants arose from the discovery of the Jones polynomial in 1984, which was soon followed by a flood of discoveries of infinitely many new invariants of knots. They were derived from interactions between knot theory and various other fields; the Jones polynomial was defined by using the theory of operator algebras, some other of the new invariants were defined by using solutions of the Yang-Baxter equation in statistical mechanics, and others still were introduced by using representations of quantum groups and by using solutions of the Knizhnik-Zamolodchikov equation, related to quantum field theory in theoretical physics, which organized them all into *quantum invariants* of knots.

Another main topic of this monograph is the study of invariants of 3-manifolds.

Until the 19th century 3-dimensional spaces appearing in mathematics and physics were mainly straight spaces (copies of the Euclidean space), but, since the beginning of the 20th century, when the theory of relativity disclosed that the real space is a globally curved space (possibly having a different topological type than the Euclidean space in the astronomical scale), curved spaces (manifolds) have become fundamental in the study of mathematics and physics of the 20th century. Furthermore, from the mid-20th-century solution of the classification problem of higher-dimensional ($\dim \geq 5$) manifolds (an historic achievement of differential topology), up to the present day, the study of low-dimensional manifolds has been the main stream of manifold theory.

Quantum theory, another great physical theory initiated at the beginning of the 20th century, asserts that an atomic particle moving in a space can be described in mathematics, not by a moving point, but by a wave function on the whole of the space. In the 1980s, various marvellous results were yielded to low-dimensional topology by the new methodology, which studies topological properties of a given space by analyzing the structure of “the space of wave functions on the underlying space” in a suitably defined quantum field theory (*i.e.*, the space of solutions of a certain differential equation, which is canonical in some physical sense). In particular in the late 1980s Witten considered a quantum field theory which depends only on the topological type of an underlying 3-manifold, and proposed that the partition function of the theory provides a “topological invariant” of the 3-manifold, where by a *topological invariant* we mean a map of the set of topological types of 3-manifolds to a well-known set. Motivated by Witten’s proposal, infinitely many invariants of 3-manifolds, called *quantum invariants*, have been constructed rigorously in various mathematical ways.

So, in the 1980s we obtained infinitely many new invariants (quantum invariants)

of knots and 3-manifolds. For these quantum invariants to be useful they needed to be organized, which was achieved with the introduction of the Kontsevich invariant (resp. the LMO invariant) and the theory of Vassiliev invariants (resp. finite type invariants) in the 1990s. Further, the Kontsevich invariant (resp. the LMO invariant) gives a map of the set of knots (resp. integral homology 3-spheres) into a lattice of infinite rank. It is now conjectured that the image under the map classifies knots (resp. integral homology 3-spheres). The set of knots (resp. integral homology 3-spheres) would be identified with a subset of the lattice if this conjecture was true.

In "classical" topology, where the study focuses, in the main, on individual knots (resp. 3-manifolds), an invariant was often regarded just as a tool to distinguish knots (resp. 3-manifolds). This situation has changed since the discoveries of these many new invariants in the 1980-90s. In the sense that an invariant of knots (resp. 3-manifolds) gives a partition (*i.e.*, a rough classification) of the set of knots (resp. 3-manifolds), the study of such a wealth of invariants is, in effect, a study of these sets. In this way, we are now ready to study the *sets* of knots and 3-manifolds.

Number theory has developed, over the course of its long history, from a study of numbers to be, also, a study of the *set* of numbers, and in this way has obtained many profound results on the numbers themselves. Can knots and 3-manifolds be to topology as numbers are to number theory? That is, we expect future developments in knot theory (resp. 3-manifold theory) to arise as much from the study of the *set* of knots (resp. 3-manifolds) as from the study of the individuals of those sets themselves. If we could find appropriate structures for those sets, then they could be studied as the "spaces of knots and 3-manifolds", and invariants (such as quantum invariants) of knots and 3-manifolds could be studied by analyzing appropriate functions on those spaces. The author hopes that this monograph will serve as a first step towards such a future study of the sets of knots and 3-manifolds.

This monograph consists of two parts. The former part (Chapter 1–7) is concerned with invariants of knots and links; see Figure 0.1 for the main knot invariants discussed in this monograph and the chapters related to them respectively. The latter part (Chapter 8–11) is concerned with invariants of 3-manifolds; see Figure 0.2 for the main 3-manifold invariants discussed in this monograph and the chapters related to them respectively.

In Chapters 1–3 the Jones and Alexander polynomials, a modern and a classical invariant of knots, are constructed from various directions by introducing knot diagrams in Chapter 1, braids in Chapter 2, and tangles in Chapter 3. The constructions of them discussed in Chapters 2 and 3 reduce the definition of the invariants of knots into invariants of "elements" (elementary tangle diagrams). In particular,

Figure 0.1 Invariants of knots and related chapters of this monograph. See also Figure 7.4 for a concrete description of this figure.

an invariant of an elementary tangle diagram with a single crossing is given by an R matrix (a solution of the Yang-Baxter equation). Such a construction is general enough to yield, not only the Jones and Alexander polynomials, but also many other invariants of knots. In fact, this construction associates an invariant of knots to every R -matrix.

As shown in Chapter 4, many R matrices systematically arise from representations of ribbon Hopf algebras, whose typical examples are quantum groups. We call invariants defined by using such R matrices derived from quantum groups *quantum invariants*. They are reformulated, in Chapter 5, by using monodromy along solutions of the KZ equation.

In Chapters 6 and 7 we discuss two approaches to control these infinitely many quantum invariants. One approach is to unify all quantum invariants into the Kontsevich invariant, which is introduced in Chapter 6 by using monodromy along solutions of the “universal” KZ equation. By definition the Kontsevich invariant is universal among quantum invariants. The other approach is to characterize all quantum invariants with a property. Vassiliev invariants are defined by such a characteristic property in Chapter 7. With the introduction of the Kontsevich invariant and the theory of Vassiliev invariants, the quantum invariants become well-organized. Further, via a descending series of equivalence relations among

Figure 0.2 Invariants of 3-manifolds and related chapters of this monograph. See also Figure 11.3 for a concrete description of this figure.

knots given by Vassiliev invariants, the Kontsevich invariant gives a map of the set of knots into a lattice of infinite rank, which is now conjectured to classify knots.

Quantum invariants of 3-manifolds were originally proposed by Witten by the Chern-Simons path integral in mathematical physics (see Appendix F). In Chapter 8 we mathematically define quantum invariants of a 3-manifold to be appropriate linear sums of quantum invariants of a framed link, which gives a surgery presentation of the 3-manifold; such a definition was predicted by the operator formalism of the Chern-Simons path integral. Further, motivated by the perturbative expansion of the Chern-Simons path integral, we define a perturbative invariant to be a certain number-theoretical limit of quantum invariants in Chapter 9. This is a power series invariant, while quantum invariants of 3-manifolds are $\mathbb{C}$ -valued.

In Chapters 10 and 11 we introduce the LMO invariant and finite type invariants which play similar roles among the perturbative invariants as the Kontsevich invariant and Vassiliev invariants play among knot invariants. In Chapter 10 we define the LMO invariant by picking up information in the Kontsevich invariant which is invariant under the Kirby moves. We also show the universality of the LMO invariant among perturbative invariants. Further, these arguments are reviewed by using a formal Gaussian integral, called the Aarhus integral. In Chapter 11 we introduce finite type invariants of integral homology 3-spheres, and show that they satisfy similar properties as those of Vassiliev invariants. Further, via a descending series

of equivalence relations among integral homology 3-spheres given by finite type invariants, the LMO invariant gives a map of the set of integral homology 3-spheres into a lattice of infinite rank, which is now conjectured to classify integral homology 3-spheres.

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Tomotada Ohtsuki
Tokyo, October 2001

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