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INTEGRABLE MODELS

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**To
My Brothers and Sisters,
Especially,
To the Memory of
Nilu.**

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PREFACE

The subject of integrable models - both classical and quantum - is fascinating. Decades of research in this area has led to mathematical developments which are quite beautiful and which unify various aspects of physical problems that appear to be disparate. These special developments, however, have not yet been widely appreciated. The primary reason for this appears to be the technical nature of the subject. The purpose of this book, therefore, is to present the basic ideas in a systematic manner so that people in different areas may share the excitement and find the methods useful in their respective areas of research.

This book grew out of a graduate course which I taught in the Fall of '88 in the Department of Physics and Astronomy at the University of Rochester. In fact, except for the last few chapters, the material in this book is exactly what was presented in the class. Consequently, derivations of various equations appear in explicit detail and the reader is free to concentrate only on the logical development of the ideas. Furthermore, most of the techniques are developed with specific examples in order to make ideas more transparent. The first eight chapters of the book develop various ideas of classical integrability with the example of the Korteweg-de Vries equation. The geometrical approach as well as the group theoretical approach to integrable models are explained with the Toda lattice as an example. Finally, the methods of zero curvature and quantum inverse scattering are discussed within the context of the nonlinear Schrödinger equation. The level of discussion has been carefully chosen so as to make the material accessible to graduate students.

All the models I have considered are defined on the real line. Lack of space did not permit me to discuss the

periodic systems although I have tried to comment, wherever possible, on the generalization of various formulae to such cases. I have also not been able to discuss either the method of Hirota or the supersymmetric systems. Every chapter is supplemented with a list of references which I found useful in the preparation of the lecture notes. They are by no means meant to be a complete set.

There are several people whose help has been invaluable in the preparation of this manuscript. First of all, I would like to thank all the students in the class for providing a stimulating atmosphere and I would like to thank, in particular, Messrs. T. Blum, S. Borzillary and S. Ramaswamy for various forms of assistance during the preparation of the notes. I gratefully acknowledge the encouragement and support of Prof. P. Slattery. I have benefited enormously from discussions with Profs. J. Maharana, V. S. Mathur and C. Taylor and appreciate their comments and suggestions. I am truly indebted to Prof. S. Okubo who has painstakingly read through the manuscript and has generously clarified many ideas. It is also a pleasure to thank Prof. A. C. Melissinos for sharing with me his enthusiasm which has helped me through the long hours during the preparation of the manuscript. An Outstanding Junior Investigator Award from the U.S. Department of Energy during the period when this book was written is also gratefully acknowledged.

The credit for the get up of the book goes completely to Judy Mack who has gracefully typed the manuscript and has accommodated all the changes in a very short time. And finally, I would like to express my heartfelt gratitude to Ammani whose unending desire for perfection has been a constant source of inspiration to me.

Ashok Das

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