LECTURES ON GROUPS AND VECTOR SPACES

For Physicists

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LECTURES ON GROUPS AND VECTOR SPACES

For Physicists

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PREFACE

The theory of groups and vector spaces has many important applications in a number of branches of modern theoretical physics. These include the formal theory of classical mechanics, special and general relativity, solid state physics, general quantum theory, and modern elementary particle physics. For this reason, the Theoretical Physics Option of the final year BSc. physics programme at Imperial College has for many years included a course entitled “Algebra and Groups” and these notes are the contents of the course as given by the author during the last few years.

Naturally, one aim of the course is to provide an introduction to the general theory of groups and vector spaces, with particular emphasis being placed on the potential applications of the mathematical ideas to the structural foundations of quantum theory.

However another, and equally important, aim of the course is to introduce the student to the art of genuine ‘mathematical’ thinking. The physics students at Imperial College attend several ancillary mathematics courses as part of their undergraduate studies, but these are almost entirely concerned with so-called ‘mathematical methods’ (special functions, complex analysis, matrices, three-dimensional vector calculus etc.) and, being taken

by the entire undergraduate class, are presented in a way that tends to minimize the underlying mathematical structure *per se*. The ‘Algebra and Groups’ course is intended to partially remedy this situation for those students engaged in the Theory Option, and this is reflected in the notes which are written in a more precise mathematical style than is often the case in courses aimed at physics students. Quite apart from the general educational value of such an exposure to abstract thinking, it is also the case that much modern theoretical physics draws on sophisticated ideas from pure mathematics. It therefore seems most important that a prospective graduate student can approach these subjects without experiencing a total shock at a too sudden exposure to such an alien culture as pure mathematics!

The course is divided into three parts. The first is a short introduction to general group theory, with particular emphasis being placed on the matrix Lie groups that play such a crucial role in modern theoretical physics. The second part deals with the theory of vector spaces, and in particular the theory of Hilbert spaces and the analytical techniques that are needed to handle the infinite-dimensional situation. The last part of the course is a short introduction to the theory of group representations and the associated theory of characters.

Finally, let me emphasise that what is presented here are just the *notes* as handed out to the students. In the context of the course, they are naturally reinforced by my verbal comments. I must apologize if the reader finds that the absence of these expository remarks renders the written material rather pithy.

CONTENTS

Preface	v
1. Groups	
1.1. <i>Monoids</i>	1
1.2. <i>The Basic Idea of a Group</i>	5
1.3. <i>Continuous Groups</i>	20
1.4. <i>Group Operations on a Set</i>	34
1.5. <i>Normal Subgroups</i>	50
2. Vector Spaces	
2.1. <i>Basic Definitions</i>	57
2.2. <i>Normed Vector Spaces</i>	67
2.3. <i>Scalar Products</i>	73
2.4. <i>Linear Operators</i>	91
2.5. <i>Unitary Operators</i>	99
2.6. <i>The Use of Unitary Operators in Quantum Theory</i>	105
3. Group Representations	
3.1. <i>Basic Definitions</i>	113
3.2. <i>Symmetry Transformations in Quantum Theory</i> .	119
3.3. <i>Reducibility of Representations</i>	124
3.4. <i>Irreducible Representations</i>	133
3.5. <i>Group Characters</i>	142
3.6. <i>The Group Algebra</i>	148
Problems	159
Solutions	169
Index	217