ANALYTICAL --- DYNAMICS

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ANALYTICAL DYNAMICS

Course Notes

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Preface and Acknowledgements

This book comprises a set of notes on a course of lectures given by the late Professor Shirley L. Quimby at Columbia University, New York City, and is dedicated to his memory. The notes contain proofs of basic theorems, derivations of formulae, and amplification of observations, as well as the presentation and solution of illustrative problems.

Prof. Quimby will be remembered both as lecturer and thesis advisor. He was demanding in his courses as well as in the laboratory, where he brought to bear respect for equipment and where he emphasized the importance and need for experimental physicists to develop all aspects of research, including theoretical understanding, experimental design, machine shop parts production, system assembly, and analysis of results.

Prof. Quimby was a physicist and educator in the true classical spirit, one too rarely seen today.

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Contents

Preface and Acknowledgements	v
Contents	vii
Introductory note and references	xiii

Chapter I Kinematics of a Rigid Body

A. Finite Displacements	1
1. Specification of position; planar displacement	2
2. Specification of translational displacement	4
3. Specification of rotational displacement	4
4. Rotation about a fixed axis: Rodrigue's formula	4
5. Generalization of Rodrigue's formula	5
6. Euler's theorem	6
7. Application to coordinate systems	7
8. Theorem: Resultant displacement of two or more rotations	11
9. Theorem: Planar displacement as a rotation about a certain axis	12
10. Theorem: Rotation followed by a translation	14
11. Theorem: Displacement as a translation plus a rotation	15
12. Chasle's theorem	16
13. Analytical representation of a general displacement	19
14. The Eulerian angles	20
15. Review of matrix algebra	21
16. Matrix for transformation of coordinates from fixed X, Y, Z -axes to Eulerian x, y, z -axes	24
B. Infinitesimal Displacements	26
17. Theorem: Velocities of points in a rigid body with one point fixed	26
18. Theorem: Resultant of simultaneous angular velocities	27
19. Analogy between angular velocities and static forces	28
20. Composition of angular velocities about parallel axes	28
21. Composition of angular velocities about skew axes	29
22. Mozzi's theorem	31
23. Euler's kinematical equations	35

Chapter II Kinematics of a Particle

24. Time rates of change of a vector in fixed and rotating coordinate systems	37
25. Relation between velocity and acceleration of a moving point in fixed and rotating coordinate systems	38
26. Velocity and acceleration of a moving point in polar coordinate systems	39
27. Velocity and acceleration of a moving point in arc coordinates	41

Chapter III Dynamics of Systems of Particles

A. The Newtonian Method	42
28. Inertial reference frame	42
29. Newtonian formulation	44
30. Secondary inertial reference system	47
31. Motion of a particle under central forces	48
32. Motion of two particles when the mutual force is directed along the line joining them	53
B. The Lagrangian Method	55
33. Holonomic and non-holonomic constraints; degrees of freedom; virtual displacement	55
34. Generalized coordinates	60
35. Kinetic energy of a system	60
36. Lagrange's equations of motion	62
37. Potential energy of a system; conservative systems	65
38. Examples of the Lagrangian method	66
39. Lagrange's method of undetermined multipliers	74
40. Evaluation of constraint forces for non-holonomic systems	79
41. Solution of equations of motion	80
42. Methods of solution by quadratures I	82
43. Significance of integrals obtained by ignorance of coordinates	86
44. Euler's theorem for homogeneous functions	88
45. The integral of energy for conservative systems	89
46. Methods of solution by quadratures II	90
47. Methods of solution by quadratures III	97
48. Methods of solution by quadratures IV	101
49. Methods of solution by quadratures V	104
50. Methods of solution by quadratures VI	105

Chapter IV Dynamics of a System of Rigid Bodies

A. General Dynamical Relations for an Aggregate of Mass Particles	106
51. Principle of linear momentum	106
52. Principle of angular momentum	106
53. Principle of independence of motion of and about the centroid	107
B. Dynamical Specification of a Rigid Body	109
54. Moment of inertia of an aggregate of mass particles	109
55. Poinot ellipsoid of inertia	109
56. Nonian form of Φ_b	110
57. Normal form of Φ_b	110
58. The ellipsoid of gyration: $\bar{\mathbf{p}}' \cdot M \Phi_b^{-1} \cdot \bar{\mathbf{p}}' = 1$	111
59. Theorems concerning moments of inertia	112
60. Angular momentum $\mathbf{J}$ of a rigid body moving with one point fixed	115
61. Angular momentum of a rigid body with respect to motion referred to a reference frame	116
62. Kinetic energy of a rigid body moving with one point fixed	117
63. Kinetic energy of a rigid body with respect to motion referred to a reference frame	117
64. Relation between kinetic energy and angular momentum	117
C. Problems of Rigid Dynamics	118
65. Systems with one degree of freedom (motion about a fixed axis)	118
66. Systems with two degrees of freedom	119
67. Systems with three degrees of freedom	119
68. Poinot analysis	119
69. The polhode cone	120
70. The polhode curve	122
71. The herpolhode curve	123
72. Free axes: stability of motion	123
73. Motion when two principal moments of inertia are equal	125
74. Euler's dynamical equations of motion of a body moving with one point fixed and of a free body referred to the centroid	128
75. Integration of Euler's dynamical equations for torque-free motion when $A > B > C$	132
76. Special case of integration of Euler's dynamical equations	136
77. Motion under gravity of a rigid body, one of whose points is fixed	137

78. Symmetrical top	137
79. Gyroscopic phenomena	144
D. Impulsive Motion	146
80. Impulsive motion	146
81. Work done by impulsive forces and torques	146
82. Examples of impulsive motion I	147
83. Examples of impulsive motion II	149
84. Examples of impulsive motion III	150
85. Lagrange's equations for impulsive motion	152
E. Dissipative Motion	154
86. Dissipative motion defined	154
87. Friction	155
88. Impact	158
89. Dissipative resistive force with velocity dependence	165

Chapter V Theory of Small Vibrations

A. Matrices, Linear Transformations and Quadratic Forms	168
90. Matrices and homogeneous quadratic forms	168
B. Free Vibrations of Dissipative Systems	183
91. Stable equilibrium	183
92. Free motion in the neighborhood of equilibrium	184
93. Properties of the three quadratic forms	185
94. Lagrangian equations of motion	187
95. Roots of the secular equation	188
96. Case of T and $V \neq 0$ and $\mathcal{F}$ not too large	190
97. Vibratory motion of a not too dissipative system	191
C. Free Vibrations of Conservative Systems	195
98. Normal coordinates	195
99. Principal modes of vibration	197
100. Preliminary transformation of coordinates	198
101. Geometrical interpretation of solution of vibration problem I	203
102. Geometrical interpretation of solution of vibration problem II	208
103. Solution of vibration problem — iteration method	208

104. Multiple roots of the λ equation (degeneracy)	209
105. Zero root of the λ equation	210
106. Total energy associated with normal vibrations	212
107. Effect of constraints on characteristic frequencies	212
108. Characteristic frequencies of constrained vibrations	213
109. Effect of perturbations	215
D. Normal Coordinates and Dissipative Systems	217
110. Dissipative coupling of normal modes	217
E. Forced Vibrations	218
111. Application of generalized forces	218
112. The case of dynamic displacement	220
F. Transverse Vibrations	222
113. Transverse vibrations of a string of beads	222

Chapter VI General Dynamical Systems

114. Notion of a general dynamical system	230
115. Coordinates of a general dynamical system	230
116. Hypothesis of a general dynamical system	231
117. Procedure I	231
118. Procedure II	233
119. "Energy" of a general dynamical system	238

Chapter VII Additional Principles of General Dynamical Systems

120. Background to the study of the Hamiltonian method	241
121. d'Alembert's principle	243
122. Gauss's principle of least constraint (1829).	246
123. Configuration space and configuration-time space	248
124. Review of notation and elementary operations of calculus of variations	249
125. Hamilton's principle of varying action	251
126. Hamilton's principle	252
127. Hamilton's principle for non-conservative holonomic systems	252
128. Hamilton's principle as a problem in the calculus of variations	253
129. Derivation of Lagrange's equations (scalar and vector fields) from Hamilton's principle	255

130. Principle of least action for conservative holonomic systems 260

131. Reciprocal functions and the Legendre transformation 264

132. Hamilton’s canonical equations 266

Chapter VIII The Hamiltonian Method in Dynamics

133. Remarks concerning the Hamiltonian method 271

134. Independent variables for the description of the motion of dynamical systems .. 271

135. Solution of Hamilton’s equations when none of the coordinates or the time
appears explicitly in the Hamiltonian 276

136. Derivation of the canonical (contact) transformation from Hamilton’s
principle and the Legendre transformation 276

137. Examples of canonical transformations 279

138. Alternative definition of a canonical transformation 281

139. Alternative method of attack on problems of mechanics 282

140. Hamilton’s integral as a solution of the Hamilton–Jacobi equation 286

141. Methods for integrating the Hamilton–Jacobi equation I 287

142. Methods for integrating the Hamilton–Jacobi equation II 289

143. Methods for integrating the Hamilton–Jacobi equation III 293

144. Separation of variables for Liouville –Type *T* and *V* 295

145. Connection between quantum mechanics and classical mechanics 298

Introductory Note

Some collateral reading is indicated at appropriate places in the text, but the student should pursue his study far beyond these brief references. Acquaintance with the literature is one of the most important benefits to be derived from this as from almost any course.

The key to the references is as follows: (Wi-S-72) means Wills, Section 72; (W-C-5) means Whittaker, Chapter 5; and (KB-P-238) means Karman and Biot, page 238.

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- Ma Mach, E. *Science of Mechanics*
- MF Morse, P.M. and Feshbach, H. *Methods of Theoretical Physics*
- MM Margenau, H. and Murphy, G. *Mathematics of Physics and Chemistry*

O	Osgood, W.F. Advanced Calculus
Ol	Olson, H.F. Dynamical Analogies
P	Page, L. and Adams, N.I. Electrodynamics
Po	Poincare, H. Science and Hypothesis
R	Rayleigh, Lord Theory of Sound
Ro	Rojansky, V. Introductory Quantum Mechanics
Rou	Routh, E.J. Elementary Rigid Dynamics
Rout	Routh, E.J. Advanced Rigid Dynamics
S	Seitz, F. Modern Theory of Solids
SF	Slater, J.C. and Frank, N.H. Mechanics
SG	Synge, J.L. and Griffith, B.A. Principles of Mechanics
So	Sommerfeld, A. Atomic Structure and Spectral Lines
St	Stratton, J.A. Electromagnetic Theory
Sw	Swann, W.F.G. Reviews of Modern Physics, v. 2, No. 3, July 1930.
Swa	Swann, W.F.G. Bulletin National Research Council, No. 24
T	Thomson, J.J. Applications of Dynamics to Physics and Chemistry
W	Whittaker, E.T. Analytical Dynamics
We	Webster, A.G. The Dynamics of Particles, and of Rigid, Elastic and Fluid Bodies
Web	Webster, A.G. Partial Differential Equations of Mathematical Physics
Webs	Webster, A.G. The Theory of Electricity and Magnetism
Wi	Wills, A.P. Vector and Tensor Analysis

The following excellent standard treatises may be noted in addition to the above:

Thomson and Tait Treatise on Natural Philosophy (2 volumes)

Klein and Sommerfeld Theorie der Kreisels (4 volumes)