PHASE SPACE PICTURE OF QUANTUM MECHANICS

Group Theoretical Approach

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PHASE SPACE PICTURE OF QUANTUM MECHANICS

Group Theoretical Approach

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PREFACE

Quantum mechanics can take different forms. The Schrödinger picture of quantum mechanics is very useful in atomic and nuclear physics. The Heisenberg picture is the basic language for the covariant formulation of quantum field theory. Is there then any need for a new picture of quantum mechanics? This depends on whether there are branches of physics where the Schrödinger or Heisenberg picture is less than fully effective.

Quantum optics and relativistic bound-state problems are relatively new fields. In quantum optics, we deal with creation and annihilation of photons and linear superposition of multiphoton states. It is possible to construct the mathematics of harmonic oscillators in the Schrödinger picture to describe the photon's states. However, the mathematics becomes complicated when we attempt to describe generalized coherent states often called the squeezed states. Is there a language simpler than the Schrödinger picture?

Quantum field theory accommodates both the uncertainty principle and special relativity. However, it is less than fully effective in describing bound-state problems or localized probability distributions. It is possible to construct models of relativistic hadrons consisting of quarks starting from the Schrödinger picture of quantum mechanics. The question then is whether it is possible to formulate the uncertainty relations in a covariant manner (Dirac 1927).

The phase-space picture of quantum mechanics provides the answer to these questions. Starting from the Schrödinger wave function, it is possible to construct a distribution function, often called the Wigner function, in phase space in terms of the *c*-number position and momentum variables. In this picture, it is possible to perform canonical transformations as in the case of classical mechanics. This will bring us a deeper understanding of the uncertainty principle.

This phase-space picture of quantum mechanics is not new. The earliest application of the Wigner phase-space distribution function was made in quantum corrections to thermodynamics in 1932 (Wigner 1932a). Since then, the Wigner function has been discussed in many branches of physics including statistical mechanics, nuclear physics, atomic and molecular physics, and foundations of physics. However, it is difficult to see the advantage of using the Wigner function over the existing method in those traditional branches of physics.

In this book, we discuss applications of the Wigner function in quantum optics and the relativistic quark model which are relatively new subjects in physics and

which still need a basic scientific language. From the mathematical point of view, the Wigner function for the ground-state harmonic oscillator is the basic language for these new branches of physics. However, its symmetry properties constitute the most interesting aspect of this new scientific language.

Indeed, the symmetry property of the Wigner function in phase space is that of the Lorentz group. The Lorentz group is known to be a difficult subject to mathematicians, because it is a non-compact group. To physicists, group theory is a difficult subject when its representations have no physical applications. However, the situation is quite the opposite when the representation can extract physical implications.

In this book, we discuss the physical consequences of the symmetries of the Wigner function in phase space. This book is written for those scientists and students who wish to study the basic principles of the phase-space picture of quantum mechanics and physical applications of the Wigner distribution functions. This book will also serve a useful purpose for those who simply wish to study the physical applications of the Lorentz group.

We are indebted to Professor Eugene P. Wigner for encouraging us to formulate a group theoretical approach to the phase-space picture of quantum mechanics. Professor Wigner suggested the use of the light-cone coordinate system for the covariant formulation of the Wigner function. Indeed, Chapter 10 of this book is based on Professor Wigner's ideas. He suggested the possibility that the work of Inonu and Wigner (1953) on group contractions be extended to study the space-time geometry of relativistic particles (Kim and Wigner 1987a and 1990a). He also suggested the use of the concept of entropy when the measurement process is less than complete in a relativistic system (Kim and Wigner 1990c).

While this book was being written, we received helpful comments and suggestions from many of our colleagues, including K. Cho, D. Han, C. H. Kim, M. Kruger, P. McGrath, H. S. Pilloff, L. Rana, Y. H. Shih, J. Soln, C. Van Hine, and W. W. Zachary.

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INTRODUCTION

The concept of phase space arises naturally from the Hamiltonian formulation of classical mechanics, and plays an important role in the transition from classical physics to quantum theory. However, in quantum mechanics, the position and momentum variables cannot be measured simultaneously. In the Schrödinger picture, the wave function is written as a function of either the position or the momentum variable, but not of both. For this reason, in quantum mechanics, the density matrix (Von Neumann 1927 and 1955) replaces phase space as a device for describing the density of states. It therefore appears that phase space is not a useful concept in quantum mechanics. We disagree. The role of phase space in quantum mechanics has not yet been fully explored.

Starting from the density matrix, is it possible to develop an algorithm of quantum mechanics based on phase space? This question has been raised repeatedly since the publication in 1932 of Wigner's paper on the quantum correction for thermodynamic equilibrium (Wigner 1932a). Since it is not possible to measure simultaneously position and momentum without error, it is meaningless to define a point in phase space. However, this does not prevent us from defining an area element in phase space whose size is not smaller than Planck's constant. Since the measurement problem is stated in terms of the least possible value of the product of the uncertainties in the position and momentum, it is of interest to see how the uncertainty product can be stated in phase space.

The basic advantage of this phase-space picture of quantum mechanics is that it is possible to perform canonical transformations, just as in classical mechanics. The purpose of this book is to study the physical consequences derivable from canonical transformations in quantum mechanics. Using these transformations, we can compare quantum mechanics with classical physics in terms of many illustrative examples. In addition, the phase-space picture of quantum mechanics is becoming a new scientific language for modern optics which is a rapidly expanding field. Furthermore, the Lorentz transformation in a given direction of boost is a canonical transformation in the light-cone coordinate system. This allows us to state the uncertainty relation in a Lorentz-invariant manner.

There are still many questions concerning the uncertainty relations for which answers are not well known. For instance, in the Schrödinger picture, the free-particle wave packet becomes widespread, and the uncertainty product increases as time progresses or regresses. Is it possible to state the uncertainty relation

in terms of the quantity which remains constant? Can phase space provide an answer to this question? The answer to this question is YES. In the phase space-picture, the uncertainty is defined in terms of the area which the Wigner distribution function occupies. The spread of a wave packet is an area-preserving canonical transformation in the phase-space picture of quantum mechanics.

Quantum optics is a rapidly expanding subject, and it is increasingly clear that coherent and squeezed states of light will play a major role in a new understanding of the uncertainty principle, and will provide innovations in high-technology industrial applications. These optical states are minimum-uncertainty states, and transformations among these states are therefore canonical transformations. Indeed, the phase-space picture of quantum mechanics is the natural language for these relatively new quantum states.

Most physicists these days learn classical mechanics from Goldstein's textbook (Goldstein 1980). However, Goldstein's book does not emphasize the importance of linear canonical transformations, which are discussed in more advanced books (Arnold 1978, Abraham and Marsden 1978, Guilemin and Sternberg 1984). In this book, we shall discuss the group of linear canonical transformations in phase space which is the inhomogeneous symplectic group (Han *et al.* 1988). For a single pair of canonically conjugate variables, the group is the inhomogeneous symplectic group $\text{ISp}(2)$, and it is $\text{ISp}(4)$ for two pairs of conjugate variables.

If we do not take into account translations in phase space, the symmetry groups become those of homogeneous symplectic transformations. The groups $\text{Sp}(2)$ and $\text{Sp}(4)$ are locally isomorphic to the $(2 + 1)$ -dimensional and $(3 + 2)$ -dimensional Lorentz groups. Thus the study of the symmetries in phase space requires the study of Lorentz transformations.

The Lorentz transformation is one of the most fundamental transformations in physics, and this subject can be formulated in terms of the inhomogeneous Lorentz group (Wigner 1939). Since this group governs the fundamental space-time symmetries of elementary particles, there are many papers and books on this subject (Kim and Noz 1986). In this book, we treat Lorentz transformations as canonical transformations.

One of the persisting questions in modern physics is whether the uncertainty relations can be Lorentz-transformed. Does Planck's constant remain invariant under Lorentz transformations? Is localization of the probability distribution a Lorentz-invariant concept? It is very difficult to answer these questions in the Heisenberg or Schrödinger picture of quantum mechanics. The basic limitation of these pictures is that they do not tell us how the uncertainty relations appear to observers in different Lorentz frames. The question of whether quantum mechanics can be made consistent with special relativity has been and still is the central issue of modern physics.

We shall address this question within the framework of the phase-space picture of quantum mechanics. It is interesting to note that the Lorentz boost in a given direction is a canonical transformation in phase space using the light-cone variables. This allows us to state the uncertainty relations in a Lorentz-invariant

manner. Feynman's parton picture (Feynman 1969) and the nucleon form factors are discussed as illustrative examples.

In the first two Chapters, we discuss the forms of classical mechanics and quantum mechanics useful for the formulation of the Wigner phase-space picture of quantum mechanics, which is discussed in detail in Chapters 3 and 4. Chapters 5 and 6 are for the applications of the Wigner function to coherent and squeezed states of light. It is seen in these chapters that the study of the Wigner function requires the knowledge of the Lorentz group.

In Chapters 7 and 8, we present a detailed discussion of the physical representations of the inhomogeneous Lorentz group or the Poincaré group which governs the fundamental space-time symmetries of relativistic particles. By constructing the representation based on harmonic oscillators, we study the phase-space picture of relativistic extended particles. Chapter 9 contains a detailed discussion of experimental observation of Lorentz-squeezed hadrons. Finally, in Chapter 10, we discuss some fundamental issues in space-time symmetries of relativistic system, including the unification of space-time symmetries of massive and massless particles and the entropy increase due to the incompleteness in measurements.

Since we are combining the Wigner function with group theory, we have reprinted in the Appendix Wigner's 1932 paper on the Wigner function as well as his 1939 paper on the representations of the inhomogeneous Lorentz group. The study of phase space requires a knowledge of harmonic oscillators. P.A.M. Dirac was interested in constructing representations of the Lorentz group based on four dimensional harmonic oscillators. We have therefore included Dirac's 1945 paper on the Lorentz group and his 1963 paper on the de Sitter group.

There are many other interesting subjects which can be studied within the framework of the phase-space picture of quantum mechanics but are not discussed in this book. However, there are now a number of review articles (Wigner 1971, O'Connell 1983, Carruthers and Zachariasen 1983, Hillery *et al.* 1984, Balazs and Jennings 1984, Littlejohn 1986) containing applications of the Wigner phase-space distribution function to various branches of modern physics. The scope of this book is limited to the simplest form of the Wigner function with maximum symmetry applicable to the branches of physics in which the phase-space picture is definitely superior to other forms of quantum mechanics.

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