Applied Functional Analysis and Partial Differential Equations

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Contents

Preface

1	Linear Operators in Banach Spaces	1
1.1	Metric Spaces	1
1.2	Vector Spaces	6
1.3	Banach Spaces	8
1.4	Linear Operators	11
1.5	Duals	16
1.6	Spectrum	23
1.7	Compact Linear Operators	29
1.8	Boundary Value Problems for Linear ODEs	39
1.9	Exercises	43
2	Linear Operators in Hilbert Spaces	47
2.1	Orthonormal Sets	47
2.2	Adjoint Operators	54
2.3	Accretive Operators	58
2.4	Weak Solutions	62
2.5	Example: Constant Coefficient PDEs	68
2.6	Self-adjoint Operators	70
2.7	Example: Sturm-Liouville Problem	76
2.8	Sectorial Forms	80
2.9	Example: Harmonic Oscillator and Hermite Functions	88
2.10	Example: Completeness of Bessel Functions	92
2.11	Example: Finite Element Method	96
2.12	Friedrichs Extension	99
2.13	Exercises	104
3	Sobolev Spaces	109
3.1	Introduction	109
3.2	Fourier Transform	114
3.3	Distributions	122

3.4	Weak Derivatives	126
3.5	Definition and Basic Properties of Sobolev Spaces	135
3.6	Imbeddings of $W^{m,p}(\Omega)$	140
3.7	Elliptic Problems	153
3.8	Regularity of Weak Solutions	160
3.9	Exercises	163
4	Semigroups of Linear Operators	167
4.1	Introduction	167
4.2	Bochner Integral	171
4.3	Basic Properties of Semigroups	176
4.4	Example: Wave Equation	187
4.5	Sectorial Operators and Analytic Semigroups	192
4.6	Invariant Subspaces	204
4.7	The Inhomogeneous Problem - Part I	209
4.8	Exercises	212
5	Weakly Nonlinear Evolution Equations	215
5.1	Introduction	215
5.2	Basic Theory	218
5.3	Example: Nonlinear Heat Equation	222
5.4	Approximation for Evolution Equations	224
5.5	Example: Finite Difference Method	233
5.6	Example: Galerkin Method for Parabolic Equations	236
5.7	Example: Galerkin Method for the Wave Equation	239
5.8	Friedrichs Extension and Galerkin Approximations	243
5.9	Exercises	245
6	Semilinear Parabolic Equations	247
6.1	Fractional Powers of Operators	247
6.2	The Inhomogeneous Problem - Part II	254
6.3	Global Version	257
6.4	Main Results	259
6.5	Example: Navier-Stokes Equations	267
6.6	Example: A Stability Problem	271
6.7	Example: A Classical Solution	273
6.8	Dynamical Systems	276
6.9	Example: The Chafee-Infante Problem	279
6.10	Exercises	282

Bibliography	285
List of Symbols	289
Index	291

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Preface

This book is an introduction to partial differential equations (PDEs) and the relevant functional analysis tools which PDEs require. This material is intended for second year graduate students of mathematics and is based on a course taught at Michigan State University for a number of years. The purpose of the course, and of the book, is to give students a rapid and solid research-oriented foundation in areas of PDEs, like semilinear parabolic equations, that include studies of the stability of fluid flows and, more generally, of the dynamics generated by dissipative systems, numerical PDEs, elliptic and hyperbolic PDEs, and quantum mechanics. In other words, the book gives a complete introduction to and also covers significant portions of the material presented in such classics as *Partial Differential Equations* by Avner Friedman, *Geometric Theory of Semilinear Parabolic Equations* by Dan Henry, and *Semigroups of Linear Operators and Applications to PDEs* by A. Pazy.

The need for such a book is due to the fact that in order to study PDEs one needs to know some functional analysis, which requires a thorough knowledge of real analysis (Lebesgue integral). Therefore, if real analysis is studied in the first year of graduate school, and functional analysis in the second year, the student only begins with PDEs in the third year - and may even have to re-learn functional analysis if the prior instructor ignored unbounded operators (which sometimes happens).

The reader is expected to be comfortable with the Lebesgue integral; more specifically, with the material presented in Examples 1.3.4 and 1.5.2. The Cauchy Theorem is also used in a couple of places, with the most difficult version used in (4.44). These are the only real prerequisites for the whole book. Above this level, all theorems used are proved in the text. One may, and perhaps should, skip over some of the proofs. However, they are included in case they are needed.

With regard to the writing style, all formal statements, like Theorems, contain all assumptions except for those declared at the beginning of the section in which the statement appears. This should make it easy, even for a casual reader, to figure out what is actually assumed in a given statement. There is, however, one exception. Throughout Chapter 3 it is assumed, unless otherwise specified, that Ω is an arbitrary nonempty open set in $\mathbb{R}^n$, $n \in \mathbb{N}$.

In the first two chapters functional analysis tools are developed and differential operators are studied as examples. Sturm-Liouville operators are nice examples of self-

adjoint operators with compact resolvent and are reused in Chapter 4 as generators of strongly continuous semigroups. Hörmander's treatment of weak solutions of constant coefficient PDEs is also presented early on as an example. The foundation of elliptic, parabolic and wave equations, as well as of Galerkin approximations, is given in the section on Sectorial Forms. Throughout the text, completeness of a number of orthonormal systems is proven.

The Fourier transform and its applications to constant coefficient PDEs are presented in Chapter 3. We briefly touch upon distributions and fundamental solutions, and prove the Malgrange-Ehrenpreis Theorem. Most of Chapter 3 is devoted to study of Sobolev spaces. Many sharp results concerning existence and compactness of imbeddings, as well as interpolation inequalities, are proven. These results are applied to elliptic problems in the last two sections.

The study of evolution equations begins in Chapter 4 where the semigroup theory is introduced. The Hille-Yosida Theorem for strongly continuous semigroups and Hille's construction of analytic semigroups are presented. The semigroup theory and the results of the previous chapters enable us to discuss linear parabolic and wave equations. In preparation for studies of nonlinear evolution equations, the invariant subspaces associated with the semigroups and the inhomogeneous problem are also examined.

A dynamical systems approach to weakly nonlinear evolution equations is given in Chapter 5 with a nonlinear heat equation studied as an example. Trotter's approximation theory is adapted to such equations giving convergence of Galerkin and finite difference type approximations.

The chapter on semilinear parabolic equations begins with a very technical section on fractional powers of operators. Our main results contain existence, uniqueness, continuous dependence, maximal interval of existence, stability and instability results. These results are applied to the Navier-Stokes equations, to a stability problem in fluid mechanics, to showing how a classical solution can be obtained, and to the Chafee-Infante problem as an example of a gradient system.

I wish to thank S. N. Chow and D. R. Dunninger for their early encouragement and my wife Pam for checking the grammar.