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Preface

This volume of papers is devoted to knots and their applications. From the outset, knot theory has been an applied science with roots in weaving and in practical knot tying. The theory of knots and links in three-dimensional space is a branch of pure mathematics that is a good model of the behaviour of actual rope in the three space of our direct experience.

This theory received its inception in the 19th century with the help of Lord Kelvin (William Thomson) who, inspired by his theory of vortex atoms, asked the mathematicians Kirkwood, Little and Tait to compile tables of topologically distinct knots. These tables were an impetus to the newly growing field of topology, and by the 1920's there was a genuine mathematical theory of knots with its own special characteristics.

Along with this initial mathematical theory came an isolation of the mathematics of knots from potential applications. This isolation was partly due to the collapse of the ether theory upon which Lord Kelvin's vortices were based, and partly upon the new and abstract nature of the topological methods that were being applied to knots. Nevertheless, as time went on applications and ideas for applications came forth. In 1984 Vaughan Jones discovered a new and very simply defined invariant of knots and links. The Jones polynomial is intimately related to mathematical physics, and this new circumstance changed the entire landscape of both knot theory and its applications. This book is a collection of research articles on knots and their applications that span the entire time period from Lord Kelvin to the present day.

Here is a brief description of the contents of this volume.

The first article, "Knot Logic" by Louis Kauffman, is a research article written for this volume. Here the application of knot theory is to the foundations of mathematics. The technique of knot diagrams is used to give non-standard models for set theory and this is generalized to include a diagrammatic version of the Church-Curry lambda calculus and its formal relationship with invariants of knots and links (the quandle, crystal and rack). One advantage of this approach via the lambda calculus is that it allows a generalization of the rack to an interlock algebra that is a complete knot invariant. Many examples are given that relate paradoxes of self-reference and puzzles (such as the "Chinese rings" — see the example of recursive unlinking in section 2) to the theory of knots. Other topics in this essay include an exposition of recent work of Laver and DeHornoy applying the Artin braid group to the solution of the word problem for certain algebras arising in set theory, relations between switching circuit theory, electrical networks and knot theory and a look at the formalism of self-reproducing machines in knot diagrammatic terms. The essay ends

with a discussion of the idea of knots and links as the "pregeometry" desired by John Wheeler [CTW] for foundations of physics. An appendix outlines the bracket construction for the Jones polynomial.

Other than brief mention of the bracket polynomial and related matters in the Knot Logic paper, this volume does not directly address the interplay between knot theory and statistical mechanics that has ensued since the discovery of the Jones polynomial. The interested reader should consult [K1],[K2],[K3],[J],[W],[Wu] for further information on this topic.

Next comes a reprinting of three articles by Lord Kelvin on the theory of vortex atoms. These are entitled

1. "On Vortex Atoms"
2. "On Vortex Motion"
3. "Vortex Statics".

The first and third articles are qualitative in nature, and show very clearly Kelvin's physical mode of thinking. The middle article is more mathematical, and will give the reader a good flavor of his mode of analysis in relation to these problems. The idea of a knotted vortex is very attractive and we shall see it recur in later papers in this book.

The papers by Kelvin were written between 1867 and 1880. We skip, for the next paper, to a reprinting of the 1968 paper by David Finkelstein and Julio Rubenstein entitled "Connection between Spin, Statistics, and Kinks". This paper uses knots (and a trick with a rubber band that you can perform on your fingers) to analyse a modern problem involving spin in quantum mechanical systems.

Next we reprint a remarkable 1972 paper by Herbert Jehle entitled "Flux Quantization and Particle Physics". Jehle made the hypothesis that elementary physical particles are quantized electromagnetic flux loops. In his theory a particle can be identified with a knot or link representing this flux. The antiparticle for a given knot or link is its mirror image. This vision of particulate matter as trapped loops of quantum electromagnetic energy is a direct descendant of Lord Kelvin's vortices. Jehle's theory is incomplete, but a fascinating signpost in the history of attempts to apply knot theoretic ideas to basic physics.

The next article is a reprint of the 1977 paper by Eckehard W. Mielke entitled "Knot Wormholes in Geometrodynamics?". It has been suggested by John Wheeler that a particle may "really" be the entrance to a long tube leading out of observable space and returning elsewhere (possibly in another universe) as a paired anti-particle. Such an entity is called a wormhole. Mielke considers knotted wormholes in the context of a theory of general relativity with two spatial dimensions and one temporal dimension. Here again the idea of the knotted vortex or flux tube has taken hold in a new way in a modern physical context.

We follow with a recent (1992) article by H.K. Moffatt and Renzo L. Ricca entitled "Helicity and the Calugareanu Invariant". This article is a rigorous application of topological ideas to flux tubes in the context of fluid dynamics. It also contains interesting comments about the history of a famous relationship about twist, writhe and linking number for space curves. As Moffatt and Ricca point out, the original source for the idea of "self-linking" or writhe of a space curve is the (1959) paper of Calugareanu [C]. On the other hand, it is certainly the case that the modern point of view on this relationship should be credited to James White in his paper [Wh] of 1969. The relationship in question states that the linking number (Lk) of the two edges of a closed oriented band embedded in three-dimensional space can be decomposed as a sum of the Twist (Tw) and the Writhe (Wr) of the band. Twist measures how the band twists around its core and Writhe measures the "self-linking" of the core. The core is the space curve described by the central curve in the band. This equation, $Lk = Tw + Wr$, has been applied by White and his co-workers to a myriad of problems in molecular biology. See [K3], Part II - Section 15 for a concise description of this work.

Next is an article written for this volume by Lev Rozansky entitled "Witten's Invariant of 3-Dimensional Manifolds: Loop Expansion and Surgery Calculus". This article describes Rozansky's research in the quantum field theoretic interpretation of invariants of links and three manifolds. This approach was pioneered by Edward Witten in his 1989 paper "Quantum Field Theory and the Jones Polynomial" [W]. Rozansky's article can serve as an introduction to the intricate methods of quantum field theory — applied to the study of knots and three manifolds. Here there is an extraordinary interplay between methods of mathematical physics and problems in topology.

The paper by Maurizio Martellini and Mario Rasetti entitled "2+1 Dimensional Quantum Gravity as a Gaussian Fermionic System and the 3D-Ising Model" turns the tables, using the quantum field theoretic interpretation of link invariants to apply them to a problem in mathematical physics.

We turn to molecular biology and chemistry with three papers:

1. L. Kauffman and Y.B. Magarshak, "Vassiliev Knot Invariants and the Structure of RNA Folding".
2. Alison MacArthur, "The Entanglement Structures of Polymers".
3. David W. Walba, Timothy C. Homan, Rodney M. Richards and R. Curtis Haltiwanger, "Synthesis and Cutting "In Half" of a Molecular Mobius Strip — Applications of Low Dimensional Topology in Chemistry".

The paper by Kauffman and Magarshak describes how to create invariants of graph embeddings that correspond to molecules, such as RNA, that fold on themselves. The paper can also be read for an introduction to the concept of a Vassiliev invariant of graphs and links and its relationship with the structure of Lie algebras.

The paper by Alison MacArthur describes the use of knot theoretic techniques in modelling the properties of long chain molecules including rubber and other materials.

The paper by David Walba and his collaborators describes not only the synthesis of molecular Mobius strips, but also of knotted molecules by the "hook and ladder" technique described in the text of the paper. These are excursions in the synthetic chemistry of molecules with particular topological structures. In the case of the knotted molecule, the editor of this book had an input that bore fruit in the laboratory. I suggested the hook and ladder technique to David Walba in 1986 while visiting the University of Iowa under Jon Simon's project on Stereochemical topology. David took up the idea with great enthusiasm and it eventually worked! It is not easy to synthesize small molecules with specified topology. Walba is the original discoverer of a molecular Mobius strip. His work gave a new impetus to the study of graph embeddings as a generalization of knot theory.

The last paper, "Turning a Penrose Triangle Inside Out" by Thaddeus M. Cowan, is an application of topology to the psychology of visual perception. Cowan uses the descriptive language of braids to coordinate and invent the structure of visual illusions, "impossible figures" and their properties in the eye of the beholder.

I would like to take this opportunity to thank the many people who have contributed to this volume by their direct contributions and many pleasing discussions. In particular I would like to thank Dr. K.K. Phua of World Scientific Publishing Company for suggesting the possibility of a book of papers on knots and their applications, David Tranha, Publishing Director, Mathematical Sciences, Cambridge University Press for the permission to reproduce the papers by Lord Kelvin, Arthur Greenspoon of Mathematical Reviews for great help in proof reading the papers and my wife Diane for many discussions, including the one in 1986 when we tried out the hook and ladder with rope and tape.

Louis H. Kauffman
Chicago, July 4, 1994

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