Completely Positive Matrices

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To our families, who make our lives complete and positive.

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Preface

Completely positive matrices are special positive semidefinite matrices. Every real positive semidefinite matrix A can be decomposed as $A = BB^T$ where B is a real matrix. A positive semidefinite matrix A is completely positive if there is such a decomposition where B is entrywise nonnegative. Completely positive matrices should not be confused with the completely positive maps that are studied in operator theory. Completely positive matrices (in the sense of this book) arise, through the related concept of completely positive quadratic forms, in the theory of block designs. Notes on this and on other applications can be found in Section 2.1.

Every matrix of rank r can be written as a sum of r (but not less) rank 1 matrices. If $A = BB^T$, then A can be represented as a sum of the rank 1 matrices $\mathbf{b}_i \mathbf{b}_i^T$, where the $\mathbf{b}_i$'s are the columns of B . If $\text{rank } A = r$, it has a decomposition $A = BB^T$ where B has r columns. For this reason, the minimal number of columns in an entrywise nonnegative matrix B such that $A = BB^T$ is called the cp-rank of A .

The two main problems in the theory of completely positive matrices are:

- (a) deciding if a given matrix is completely positive.
- (b) determining the cp-rank of a given cp matrix.

These two problems are still open. In spite of that, the topic of complete positivity is quite fascinating and quite a lot can be said on the above questions. In this book we try to compose an up-to-date summary of known results.

The book consists of three chapters. The second chapter deals with complete positivity and the third with the cp-rank. The first chapter contains the needed background, including positive semidefinite matrices, non-

negative matrices, Schur complements, cones and graphs. Some of these background topics are covered in greater detail than others, in order to make the book easily accessible even to people with only a basic knowledge of linear algebra. We also expand the discussion when the techniques used can be extended to the study of completely positive matrices.

The book can be used as a reference and as a text for a graduate or advanced undergraduate course or seminar in matrix theory. An almost final version of the book was used in a course on “topics in matrix theory” at the Technion. We are very thankful to the participants of this course for valuable remarks and suggestions.

The core of the book consists of Sections 2.1–2.5 and 3.1–3.3. With an introduction based on the preliminaries chapter it may be used in a one quarter course. Additional sections can be covered in a semesterial course. The dependence relations between the various sections is given in the following diagram:

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Haifa, December 2002

Abraham Berman
Naomi Shaked-Monderer

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