Topological Methods for Set-Valued Nonlinear Analysis

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To

Our beloved professor Dr. Eneyet Ullah Tarafdar, his wife — Mrs.
Sangha Mitra, his daughter — Hashina Tarafdar and his two sons —
Abir Tarafdar and Ashique Tarafdar.

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Preface

This book is a monograph of a significant and recent publications in non-linear analysis involving set-valued mappings. A map $T : X \rightarrow 2^Y$ is said to be a set-valued mapping if for each $x \in X$, $T(x) \subset Y$.

We need analysis, topology and geometry, i.e., a mixture of these three fields, in studying the theory of set-valued mappings. There have been a significant number of publiactions in this area of research over the last 40 years. These have become possible because there are huge applications in the fileds of Physics, Biology, Control Theory, Optimization, Economics and Game Theory.

We shall cover the following topics in this book: contraction mappings, fixed point theorems, minimax inequalities, end points, variational inequalities, generalized variational inequalities, and generalized quasi-variational inequalities, equilibrium analysis in economics, best approximation and fixed point theorems, topological degree theory, and non-expansive types of mappings and fixed point theorems.

In Chapter 5, we shall present variational inequalities, quasi-variational equalities and equilibrium analysis in economics. We have applied the topological methods to study the equilibrium analysis in economics. We shall discuss them in more details in the Introduction Chapter. In Chapter 6, we shall discuss best approximation and fixed point theorems for set-valued mappings in topological vector spaces. Finally, in Chapters 7 and 8 we shall present some aspects of degree theories for set-valued mappings and non-expansive types of mappings and fixed point theorems in locally convex topological vector spaces.

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After the sudden and unfortunate death of Dr. Enayet Tarafdar in November, 2002, I continued with the project and tried to finish it with the help of my other friends and well wishers who were working in this area of research. In this direction, I would like to mention the names of Dr. George Yuan and Dr. Peter Watson who tried to help me in finishing this project with their valuable suggestons and inputs.

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Finally, my heartfelt gratitude and thanks goes to Mrs. Sangha Mitra — wife of Dr. Tarafdar, Hashina Tarafdar — daughter of Dr. Tarafdar and his two sons — Abir Tarafdar and Ashique Tarafdar and also to my wife — Mrs. Fowzia Akhter. Abir Tarafdar helped in getting the back up of the original soft copy of this project of Dr. Tarafdar. Mrs. Sangha Mitra, Hashina Tarafdar and Ashique Tarafdar helped in searching and collecting some of the documents left at their home which were very helpful for the completion of Dr. Tarafdar's unfinished project. My wife, Fowzia Akhter, helped me invaluablely in completing the typing of some chapters of this project.

Lastly, I am very happy to acknowledge that Professor Dr. Zakaullah Khan, the Previous Head of the Dept. of Mathematics of the International Islamic University in Islamabad, Pakistan — my previous work place, helped me so generously that it created an environment for me to complete the proof-reading and final revision of the manuscript of this project for submission to the World Scientific U.K. within the specific deadline of August, 2007.

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